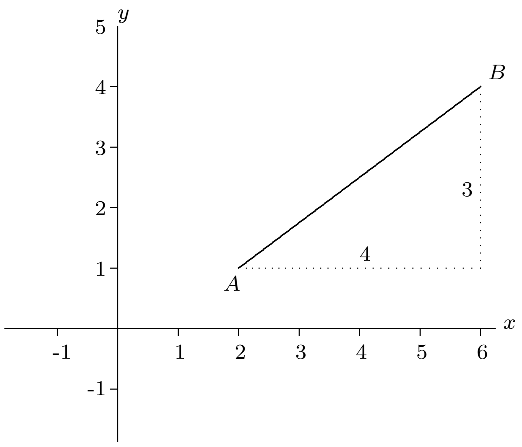
CHAPTER 2

Lines and Graphs

Almost everything in this chapter is revision from GCSE maths. It reminds you how to draw graphs, and focuses in particular on **straight line graphs** and their **gradients**. We also look at graphs of quadratic functions, and use graphs to solve equations and inequalities. An important economic application of straight line graphs is **budget constraints**.

1. The Gradient of a Line

A is a point with co-ordinates $(2, 1)$; B has co-ordinates $(6, 4)$.



When you move from A to B, the change in the x-coordinate is

$$\Delta x = 6 - 2 = 4$$

and the change in the y-coordinate is

$$\Delta y = 4 - 1 = 3$$

The gradient (or slope) of AB is Δy divided by Δx:

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{3}{4} = 0.75$$

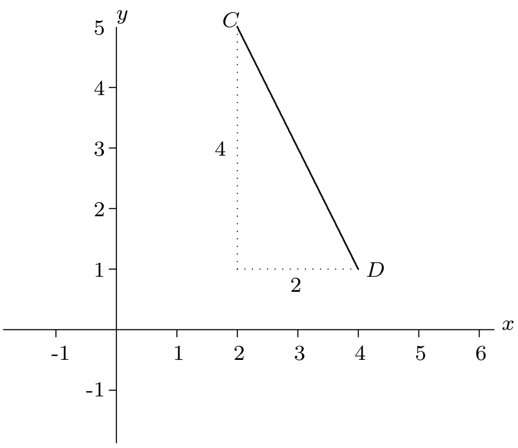
(The symbol Δ, pronounced "delta", denotes "change in".)

It doesn't matter which end of the line you start. If you move from B to A, the changes are negative, but the gradient is the same: $\Delta x = 2 - 6 = -4$ and $\Delta y = 1 - 4 = -3$, so the gradient is $(-3)/(-4) = 0.75$.

There is a general formula:

The gradient of the line joining (x_1, y_1) and (x_2, y_2) is:

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



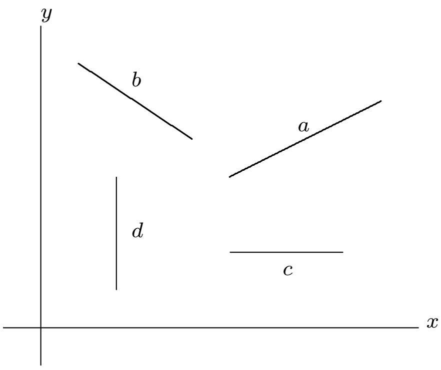
Here the gradient is negative. When you move from C to D:

$$\Delta x = 4 - 2 = 2$$

$$\Delta y = 1 - 5 = -4$$

The gradient of CD is:

$$\frac{\Delta y}{\Delta x} = \frac{-4}{2} = -2$$



In this diagram the gradient of line a is positive, and the gradient of b is negative: as you move in the x-direction, a goes uphill, but b goes downhill.

The gradient of c is zero. As you move along the line the change in the y-coordinate is zero: $\Delta y = 0$

The gradient of d is infinite. As you move along the line the change in the x-coordinate is zero (so if you tried to calculate the gradient you would be dividing by zero).

Exercises 2.1: **Gradients**

(1) Plot the points $A(1,2), B(7,10), C(-4,14), D(9,2)$ and $E(-4,-1)$ on a diagram.

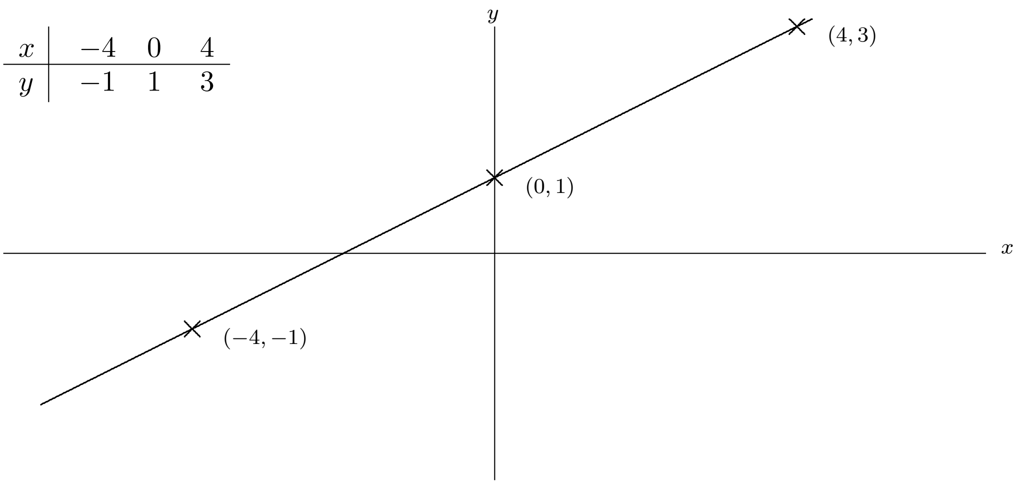
(2) Find the gradients of the lines AB, AC, CE, AD.

2. Drawing Graphs

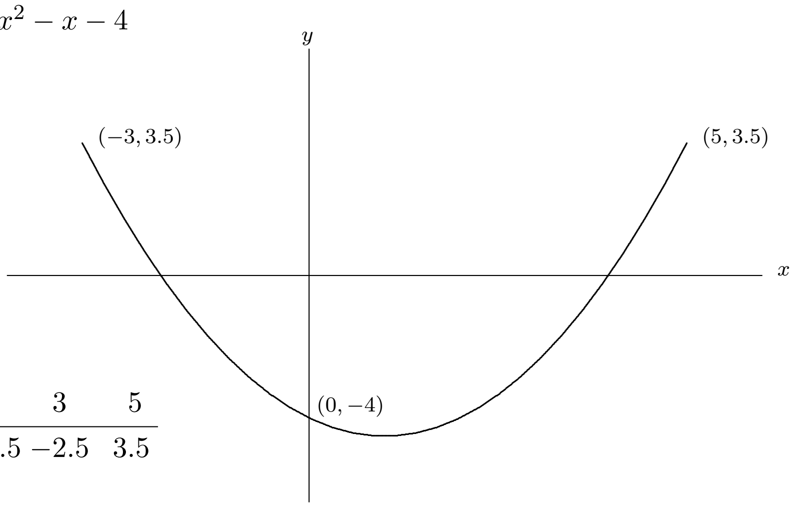
The equation $y = 0.5x+1$ expresses a relationship between 2 quantities x and y (or a formula for y in terms of x) that can be represented as a graph in x-y space. To draw the graph, calculate y for a range of values of x, then plot the points and join them with a curve or line.

Examples 2.1: $y = 0.5x + 1$

x	-4	0	4
y	-1	1	3



Examples 2.2: $y = 0.5x^2 - x - 4$



x	-3	-1	0	1	3	5
y	3.5	-2.5	-4	-4.5	-2.5	3.5

Exercises 2.2: Draw the graphs of the following relationships:

(1) $y = 3x - 2$ for values of x between -4 and $+4$.

(2) $P = 10-2Q$, for values of Q between 0 and 5. (This represents a demand function: the relationship between the market price P and the total quantity sold Q.)

(3) $y = 4/x$, for values of x between -4 and $+4$.

(4) $C = 3+2q^2$, for values of q between 0 and 4. (This represents a firm's cost function: its total costs are C if it produces a quantity q of goods.)

3. Straight Line (Linear) Graphs

EXERCISES 2.3: **Straight Line Graphs**

(1) Using a diagram with x and y axes from -4 to $+4$, draw the graphs of:

(a) $y = 2x$

(b) $2x + 3y = 6$

(c) $y = 1 - 0.5x$

(d) $y = -3$

(e) $x = 4$

(2) For each graph find (i) the gradient, and (ii) the *vertical intercept* (that is, the value of y where the line crosses the y-axis, also known as the y-intercept).

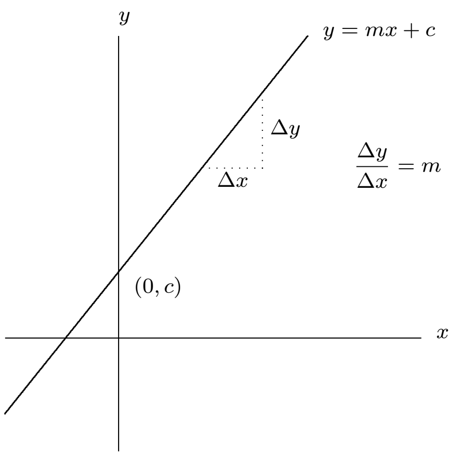
Each of the first four equations in this exercise can be rearranged to have the form $y = mx + c$:

(a) $y = 2x$	$\Rightarrow y = 2x + 0$	$\Rightarrow m = 2$	$c = 0$
(b) $2x + 3y = 6$	$\Rightarrow y = -\frac{2}{3}x + 2$	$\Rightarrow m = -\frac{2}{3}$	$c = 2$
(c) $y = 1 - 0.5x$	$\Rightarrow y = -0.5x + 1$	$\Rightarrow m = -0.5$	$c = 1$
(d) $y = -3$	$\Rightarrow y = 0x - 3$	$\Rightarrow m = 0$	$c = -3$

(e) is a special case. It cannot be written in the form $y = mx + c$, its gradient is infinite, and it has no vertical intercept.

Check these values of m and c against your answers. You should find that m is the gradient and c is the y-intercept.

Note that in an equation of the form $y = mx + c$, y is equal to a polynomial of degree 1 in x (see Chapter 1).



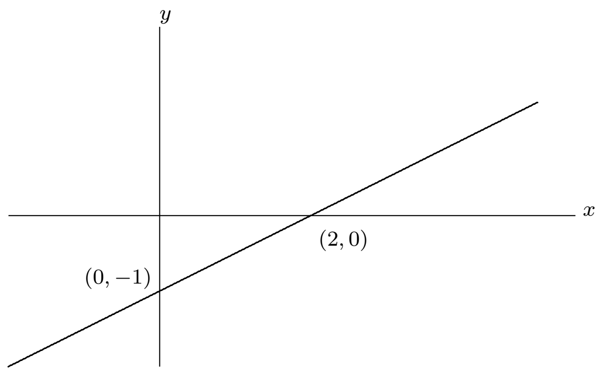
If an equation can be written in the form $y = mx + c$, then the graph is a straight line, with gradient m and vertical intercept c. We say "y is a linear function of x."

EXAMPLES 3.1: Sketch the line $x - 2y = 2$

"Sketching" a graph means drawing a picture to indicate its general shape and position, rather than plotting it accurately. First rearrange the equation:

$$y = 0.5x - 1$$

So the gradient is 0.5 and the y-intercept is -1. We can use this to sketch the graph.

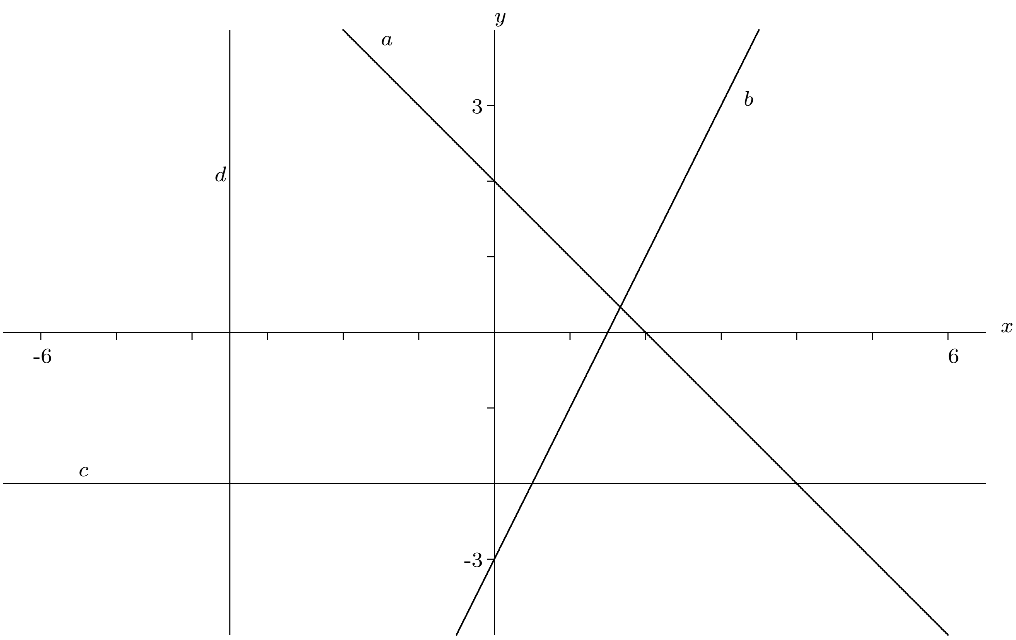


EXERCISES 2.4: $y = mx + c$

(1) For each of the lines in the diagram below, work out the gradient and hence write down the equation of the line.

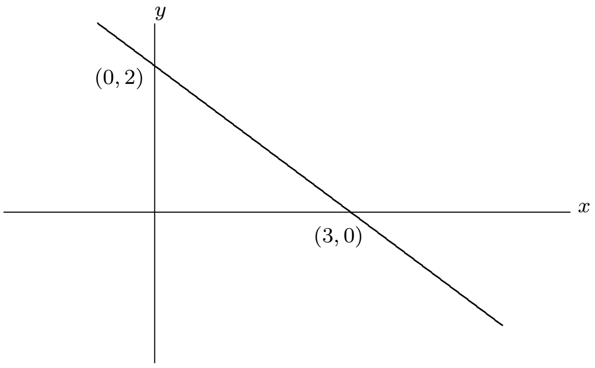
(2) By writing each of the following lines in the form $y = mx + c$, find its gradient:
(a) $y = 4 - 3x$ (b) $3x + 5y = 8$ (c) $x + 5 = 2y$ (d) $y = 7$ (e) $2x = 7y$

(3) By finding the gradient and y-intercept, sketch each of the following straight lines:
$y = 3x + 5$
$y + x = 6$
$3y + 9x = 8$
$x = 4y + 3$



3.1. Lines of the Form $ax + by = c$

Lines such as $2x+3y = 6$ can be rearranged to have the form $y = mx+c$, and hence sketched, as in the previous exercise. But it is easier in this case to work out what the line is like by finding the points where it crosses both axes.



EXAMPLES 3.2:
Sketch the line $2x + 3y = 6$.

When $x = 0, y = 2$
When $y = 0, x = 3$

EXERCISES 2.5: Sketch the following lines: (1) $4x + 5y = 100$ (2) $2y + 6x = 7$

3.2. Working out the Equation of a Line

EXAMPLES 3.3: What is the equation of the line

(i) with gradient 3, passing through $(2, 1)$?
gradient $= 3$ $\Rightarrow y = 3x + c$
$y = 1$ when $x = 2$ $\Rightarrow 1 = 6 + c$ $\Rightarrow c = -5$
The line is $y = 3x - 5$.

(ii) passing through $(-1, -1)$ and $(5, 14)$?
First work out the gradient: $\frac{14-(-1)}{5-(-1)} = \frac{15}{6} = 2.5$
gradient $= 2.5$ $\Rightarrow y = 2.5x + c$
$y = 14$ when $x = 5$ $\Rightarrow 14 = 12.5 + c \Rightarrow c = 1.5$
The line is $y = 2.5x + 1.5$ (or equivalently $2y = 5x + 3$).

There is a formula that you can use (although the method above is just as good):

The equation of a line with gradient m, passing through the point (x_1, y_1) is: $y = m(x - x_1) + y_1$

EXERCISES 2.6: Find the equations of the following lines:
(1) passing through $(4, 2)$ with gradient 7
(2) passing through $(0, 0)$ with gradient 1
(3) passing through $(-1, 0)$ with gradient -3
(4) passing through $(-3, 4)$ and parallel to the line $y + 2x = 5$
(5) passing through $(0, 0)$ and $(5, 10)$
(6) passing through $(2, 0)$ and $(8, -1)$

4. Quadratic Graphs

If we can write a relationship between x and y so that y is equal to a quadratic polynomial in x (see Chapter 1):

$$y = ax^2 + bx + c$$

where a, b and c are numbers, then we say "y is a quadratic function of x", and the graph is a parabola (a U-shape) like the one in Example 2.2.

EXERCISES 2.7: **Quadratic Graphs**

(1) Draw the graphs of (i) $y = 2x^2 - 5$ (ii) $y = -x^2 + 2x$, for values of x between -3 and $+3$.

(2) For each graph note that: if a (the coefficient of x^2) is positive, the graph is a U-shape; if a is negative then it is an inverted U-shape; the vertical intercept is given by c; and the graph is symmetric.

So, to sketch the graph of a quadratic you can:

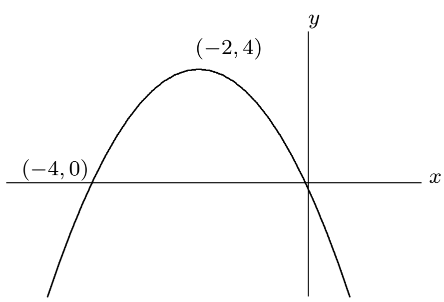
- decide whether it is a U-shape or an inverted U;
- find the y-intercept;
- find the points where it crosses the x-axis (if any), by solving $ax^2 + bx + c = 0$;
- find its maximum or minimum point using symmetry: find two points with the same y-value, then the max or min is at the x-value halfway between them.

EXAMPLES 4.1: Sketch the graph of $y = -x^2 - 4x$

- $a = -1$, so it is an inverted U-shape.
- The y-intercept is 0.
- Solving $-x^2 - 4x = 0$ to find where it crosses the x-axis:

$$x^2 + 4x = 0 \quad \Rightarrow \quad x(x + 4) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = -4$$

- Its maximum point is halfway between these two points, at $x = -2$, and at this point $y = -4 - 4(-2) = 4$.



EXERCISES 2.8: Sketch the graphs of the following quadratic functions:

(1) $y = 2x^2 - 18$

(2) $y = 4x - x^2 + 5$

5. Solving Equations and Inequalities using Graphs

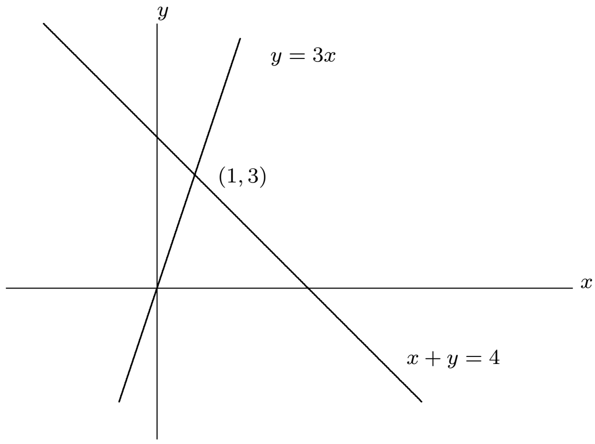
5.1. Solving Simultaneous Equations

The equations:

$$x + y = 4$$
$$y = 3x$$

could be solved algebraically (see Chapter 1). Alternatively we could draw their graphs, and find the point where they intersect.

The solution is $x = 1, y = 3$.



5.2. Solving Quadratic Equations

We could solve the solve the quadratic equation $x^2 - 5x + 2 = 0$ using the quadratic formula (see Chapter 1). Alternatively we could find an approximate solution by drawing the graph of $y = x^2 - 5x + 2$ (as accurately as possible), and finding where it crosses the x-axis (that is, finding the points where $y = 0$).

Exercises 2.9: Solving Equations using Graphs

(1) Solve the simultaneous equations $y = 4x - 7$ and $y = x - 1$ by drawing (accurately) the graphs of the two lines.

(2) By sketching their graphs, explain why you cannot solve either of the following pairs of simultaneous equations:

$$y = 3x - 5 \quad \text{and} \quad 2y - 6x = 7;$$
$$x - 5y = 4 \quad \text{and} \quad y = 0.2x - 0.8$$

(3) By sketching their graphs, show that the simultaneous equations $y = x^2$ and $y = 3x + 4$ have two solutions. Find the solutions algebraically.

(4) Show algebraically that the simultaneous equations $y = x^2 + 1$ and $y = 2x$ have only one solution. Draw the graph of $y = x^2 + 1$ and use it to show that the simultaneous equations:

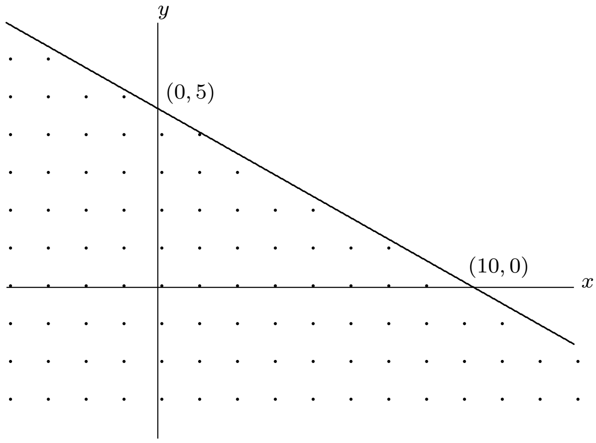
$$y = x^2 + 1 \text{ and } y = mx$$

have: no solutions if $0 < m < 2$; one solution if $m = 2$; and two solutions if $m > 2$.

(5) Sketch the graph of the quadratic function $y = x^2 - 8x$. From your sketch, find the approximate solutions to the equation $x^2 - 8x = -4$.

5.3. Representing Inequalities using Graphs

If we draw the graph of the line $2y+x=10$, all the points satisfying $2y+x<10$ lie on one side of the line. The dotted region shows the inequality $2y+x<10$.



EXERCISES 2.10: **Representing Inequalities**

(1) Draw a sketch showing all the points where $x+y<1$ and $y<x+1$ and $y>-3$.

(2) Sketch the graph of $y=3-2x^2$, and show the region where $y<3-2x^2$.

5.4. Using Graphs to Help Solve Quadratic Inequalities

In Chapter 1, we solved quadratic inequalities such as: $x^2-2x-15\leq 0$. To help do this quickly, you can sketch the graph of the quadratic polynomial $y=x^2-2x-15$.

EXAMPLES 5.1: *Quadratic Inequalities*

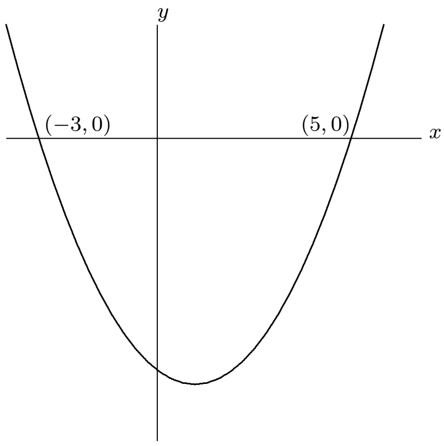
(i) Solve the inequality $x^2-2x-15\leq 0$.
As before, the first step is to factorise:

$$(x-5)(x+3)\leq 0$$

Now sketch the graph of $y=x^2-2x-15$.
We can see from the graph that $x^2-2x-15\leq 0$ when:

$$-3\leq x\leq 5$$

This is the solution of the inequality.



(ii) Solve the inequality $x^2-2x-15>0$.
From the same graph the solution is: $x>5$ or $x<-3$.

EXERCISES 2.11: Solve the inequalities:

(1) $x^2-5x>0$

(2) $3x+5-2x^2\geq 0$

(3) $x^2-3x+1\leq 0$ (Hint: you will need to use the quadratic formula for the last one.)

6. Economic Application: Budget Constraints

6.1. An Example

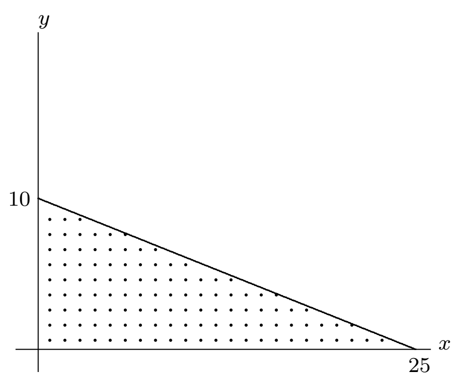
Suppose pencils cost 20p, and pens cost 50p. If a student buys x pencils and y pens, the total amount spent is:

$$20x + 50y$$

If the maximum amount he has to spend on writing implements is £5 his budget constraint is:

$$20x + 50y \leq 500$$

His budget set (the choices of pens and pencils that he can afford) can be shown as the shaded area on a diagram:



From the equation of the budget line:

$$20x + 50y = 500$$

we can see that the gradient of the budget line is:

$$-\frac{20}{50}$$

(You could rewrite the equation in the form $y = mx + c$.)

6.2. The General Case

Suppose that a consumer has a choice of two goods, good 1 and good 2. The price of good 1 is p_1 and the price of good 2 is p_2. If he buys x_1 units of good 1, and x_2 units of good 2, the total amount spent is:

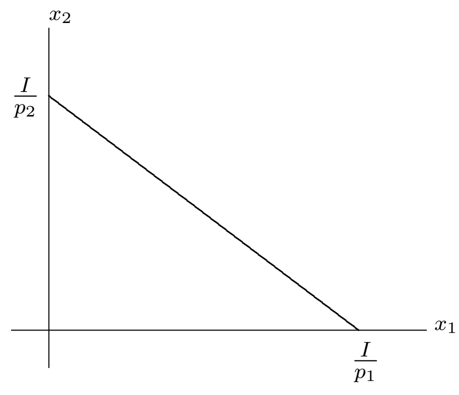
$$p_1x_1 + p_2x_2$$

If the maximum amount he has to spend is his income I his budget constraint is:

$$p_1x_1 + p_2x_2 \leq I$$

To draw the budget line:

$$p_1x_1 + p_2x_2 = I$$



note that it crosses the x_1-axis where:

$$x_2 = 0 \Rightarrow p_1x_1 = I \Rightarrow x_1 = \frac{I}{p_1}$$

and similarly for the x_2 axis.

The gradient of the budget line is:

$$-\frac{p_1}{p_2}$$

EXERCISES 2.12: **Budget Constraints**

(1) Suppose that the price of coffee is 30p and the price of tea is 25p. If a consumer has a daily budget of £1.50 for drinks, draw his budget set and find the slope of the budget line. What happens to his budget set and the slope of the budget line if his drinks budget increases to £2?

(2) A consumer has budget constraint $p_1x_1 + p_2x_2 \leq I$. Show diagrammatically what happens to the budget set if

(a) income I increases

(b) the price of good 1, p_1, increases

(c) the price of good 2, p_2, decreases.

Further reading and exercises

- *Jacques* §1.1 has more on co-ordinates and straight-line graphs.
- *Jacques* §2.1 includes graphs of quadratic functions.
- *Varian* discusses budget constraints in detail.

Solutions to Exercises in Chapter 2

Exercises 2.1:

(1)

(2) gradient (AB) $= \frac{4}{3}$,
(AC) $= -\frac{12}{5}$,
(CE) $= \infty$,
(AD) $= 0$

Exercises 2.2:

Exercises 2.3:

(1)

(2) (a) (i) 2 (ii) $y = 0$
(b) (i) $-\frac{2}{3}$ (ii) $y = 2$
(c) (i) $-\frac{1}{2}$ (ii) $y = 1$
(d) (i) 0 (ii) $y = -3$
(e) (i) ∞ (ii) None.

Exercises 2.4:

(1) (a) $y = 2 - x$
(b) $y = 2x - 3$
(c) $y = -2$
(d) $x = -3\frac{1}{2}$

(2) (a) -3
(b) $-\frac{3}{5}$
(c) $\frac{1}{2}$
(d) 0
(e) $\frac{2}{7}$

Exercises 2.5:

Exercises 2.6:

(1) $y = 7x - 26$
(2) $y = x$
(3) $y + 3x = -3$
(4) $y + 2x = -2$
(5) $y = 2x$
(6) $6y + x = 2$

Exercises 2.7:

Exercises 2.8:

Exercises 2.9:

(1) $(x, y) = (2, 1)$
(2) The lines are parallel.
(3) $(x, y) = (-1, 1)$ and $(4, 16)$
(4) $(x, y) = (1, 2)$
(5) $(x, y) \approx (0.536, -4)$ and $(7.464, -4)$ Actual values for $x = 4 \pm \sqrt{12}$

Exercises 2.10:

Exercises 2.11:

(1) $x > 5$ or $x < 0$
(2) $-1 \le x \le \frac{5}{2}$
(3) $\frac{3-\sqrt{5}}{2} \le x \le \frac{3+\sqrt{5}}{2}$

Exercises 2.12:

(1) 6 Coffee + 5 Tea $\le$ 30. Gradient (with Coffee on horiz. axis) $= -\frac{6}{5}$ Budget line shifts outwards; budget set larger; slope the same.

(2)
(a) Budget line shifts outwards; budget set larger; slope the same.
(b) Budget line pivots around intercept with x_2-axis; budget set smaller; (absolute value of) gradient increases.
(c) Budget line pivots around intercept with x_1-axis; budget set larger; (absolute value of) gradient increases.

[1]This Version of Workbook Chapter 2: August 22, 2003

Worksheet 2: Lines and Graphs

(1) Find the gradients of the lines AB, BC, and CA where A is the point $(5, 7)$, B is $(-4, 1)$ and C is $(5, -17)$.

(2) Draw (accurately), for values of x between -5 and 5, the graphs of:
(a) $y - 2.5x = -5$ (b) $z = \frac{1}{4}x^2 + \frac{1}{2}x - 1$
Use (b) to solve the equation $\frac{1}{4}x^2 + \frac{1}{2}x = 1$

(3) Find the gradient and y-intercept of the following lines:
(a) $3y = 7x - 2$ (b) $2x + 3y = 12$ (c) $y = -x$

(4) Sketch the graphs of $2P = Q + 5$, $3Q + 4P = 12$, and $P = 4$, with Q on the horizontal axis.

(5) What is the equation of the line through $(1, 1)$ and $(4, -5)$?

(6) Sketch the graph of $y = 3x - x^2 + 4$, and hence solve the inequality $3x - x^2 < -4$.

(7) Draw a diagram to represent the inequality $3x - 2y < 6$.

(8) Electricity costs 8p per unit during the daytime and 2p per unit if used at night. The quarterly charge is £10. A consumer has £50 to spend on electricity for the quarter.
   (a) What is his budget constraint?
   (b) Draw his budget set (with daytime units as "good 1" on the horizontal axis).
   (c) Is the bundle (440, 250) in his budget set?
   (d) What is the gradient of the budget line?

(9) A consumer has a choice of two goods, good 1 and good 2. The price of good 2 is 1, and the price of good 1 is p. The consumer has income M.
   (a) What is the budget constraint?
   (b) Sketch the budget set, with good 1 on the horizontal axis, assuming that $p > 1$.
   (c) What is the gradient of the budget line?
   (d) If the consumer decides to spend all his income, and buy equal amounts of the two goods, how much of each will he buy?
   (e) Show on your diagram what happens to the budget set if the price of good 1 falls by 50%.