CHAPTER 3

Sequences, Series, and Limits; the Economics of Finance

If you have done A-level maths you will have studied Sequences and Series (in particular **Arithmetic** and **Geometric** ones) before; if not you will need to work carefully through the first two sections of this chapter. Sequences and series arise in many economic applications, such as the **economics of finance and investment**. Also, they help you to understand the concept of a **limit** and the significance of **the natural number**, e. You will need both of these later.

1. Sequences and Series

1.1. Sequences

A sequence is a set of terms (or numbers) arranged in a definite order.

EXAMPLES 1.1: *Sequences*

(i) $3, 7, 11, 15, \ldots$
In this sequence each term is obtained by adding 4 to the previous term. So the next term would be 19.

(ii) $4, 9, 16, 25, \ldots$
This sequence can be rewritten as $2^2, 3^2, 4^2, 5^2, \ldots$ The next term is 6^2, or 36.

The dots($\ldots$) indicate that the sequence continues indefinitely – it is an *infinite sequence.* A sequence such as $3, 6, 9, 12$ (stopping after a finite number of terms) is a *finite sequence.* Suppose we write u_1 for the first term of a sequence, u_2 for the second and so on. There may be a formula for u_n, the n^{th} term:

EXAMPLES 1.2: *The n^{th} term of a sequence*

(i) $4, 9, 16, 25, \ldots$ The formula for the n^{th} term is $u_n = (n+1)^2$.

(ii) $u_n = 2n + 3$. The sequence given by this formula is: $5, 7, 9, 11, \ldots$

(iii) $u_n = 2^n + n$. The sequence is: $3, 6, 11, 20, \ldots$

Or there may be a formula that enables you to work out the terms of a sequence from the preceding one(s), called a *recurrence relation*:

EXAMPLES 1.3: *Recurrence Relations*

(i) Suppose we know that: $u_n = u_{n-1} + 7n$ and $u_1 = 1$.
Then we can work out that $u_2 = 1 + 7 \times 2 = 15$, $u_3 = 15 + 7 \times 3 = 36$, and so on, to find the whole sequence : $1, 15, 36, 64, \ldots$

(ii) $u_n = u_{n-1} + u_{n-2}$, $u_1 = 1$, $u_2 = 1$
The sequence defined by this formula is: $1, 1, 2, 3, 5, 8, 13, \ldots$

1.2. Series

A series is formed when the terms of a sequence are added together. The Greek letter Σ (pronounced "sigma") is used to denote "the sum of":

$$\sum_{r=1}^{n} u_r \text{ means } u_1 + u_2 + \cdots + u_n$$

EXAMPLES 1.4: *Series*

(i) In the sequence $3, 6, 9, 12, \ldots$, the sum of the first five terms is the series: $3 + 6 + 9 + 12 + 15$.

(ii) $\displaystyle\sum_{r=1}^{6}(2r+3) = 5 + 7 + 9 + 11 + 13 + 15$

(iii) $\displaystyle\sum_{r=5}^{k}\frac{1}{r^2} = \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots + \frac{1}{k^2}$

EXERCISES 3.1: **Sequences and Series**

(1) Find the next term in each of the following sequences:

(a) $2, 5, 8, 11, \ldots$
(b) $0.25, 0.75, 1.25, 1.75, 2.25, \ldots$
(c) $5, -1, -7, \ldots$
(d) $36, 18, 9, 4.5, \ldots$
(e) $1, -2, 3, -4, 5, \ldots$

(2) Find the 2^{nd}, 4^{th} and 6^{th} terms in the sequence given by: $u_n = n^2 - 10$

(3) If $u_n = \frac{u_{n-1}}{2} + 2$ and $u_1 = 4$ write down the first five terms of the sequence.

(4) If $u_n = u_{n-1}^2 + 3u_{n-1}$ and $u_3 = -2$, find the value of u_4.

(5) Find the value of $\sum_{r=1}^{4} 3r$

(6) Write out the following sums without using sigma notation:

(a) $\sum_{r=1}^{5}\frac{1}{r^2}$ (b) $\sum_{i=0}^{3} 2^i$ (c) $\sum_{j=0}^{n}(2j+1)$

(7) In the series $\sum_{i=0}^{n-1}(4i+1)$, (a) how many terms are there? (b) what is the formula for the last term?

(8) Express using the Σ notation:

(a) $1^2 + 2^2 + 3^2 + ... + 25^2$
(b) $6 + 9 + 12 + \cdots + 21$
(c) $16 + 25 + 36 + 49 + \cdots + n^2$

Further reading and exercises

- For more practice with simple sequences and series, you could use an A-level pure maths textbook.

2. Arithmetic and Geometric Sequences

2.1. Arithmetic Sequences

An *arithmetic sequence* is one in which each term can be obtained by adding a fixed number (called the *common difference*) to the previous term.

EXAMPLES 2.1: *Some Arithmetic Sequences*

(i) $1, 3, 5, 7, \ldots$ The common difference is 2.

(ii) $13, 7, 1, -5, \ldots$ The common difference is -6.

In an arithmetic sequence with first term a and common difference d, the formula for the n^{th}term is:
$u_n = a + (n-1)d$

EXAMPLES 2.2: *Arithmetic Sequences*

(i) What is the 10^{th} term of the arithmetic sequence $5, 12, 19, \ldots$?
In this sequence $a = 5$ and $d = 7$. So the 10^{th} term is: $5 + 9 \times 7 = 68$.

(ii) If an arithmetic sequence has $u_{10} = 24$ and $u_{11} = 27$, what is the first term?
The common difference, d, is 3. Using the formula for the 11th term:

$$27 = a + 10 \times 3$$

Hence the first term, a, is -3.

2.2. Arithmetic Series

When the terms in an arithmetic sequence are summed, we obtain an arithmetic series. Suppose we want to find the sum of the first 5 terms of the arithmetic sequence with first term 3 and common difference 4. We can calculate it directly:

$$S_5 = 3 + 7 + 11 + 15 + 19 = 45$$

But there is a general formula:

If an arithmetic sequence has first term a and common difference d, the sum of the first n terms is:
$S_n = \frac{n}{2}(2a + (n-1)d)$

We can check that the formula works: $S_5 = \frac{5}{2}(2 \times 3 + 4 \times 3) = 45$

2.3. To Prove the Formula for an Arithmetic Series

Write down the series in order and in reverse order, then add them together, pairing terms:

$$\begin{array}{rccccccccc} S_n = & a & + & (a+d) & + & (a+2d) & + \cdots + & (a+(n-1)d) \\ S_n = & (a+(n-1)d) & + & (a+(n-2)d) & + & (a+(n-3)d) & + \cdots + & a \\ 2S_n = & (2a+(n-1)d) & + & (2a+(n-1)d) & + & (2a+(n-1)d) & + \cdots + & (2a+(n-1)d) \\ = & n(2a+(n-1)d) \end{array}$$

Dividing by 2 gives the formula.

EXERCISES 3.2: **Arithmetic Sequences and Series**

(1) Using the notation above, find the values of a and d for the arithmetic sequences:
(a) $4, 7, 10, 13, \ldots$ (b) $-1, 2, 5, \ldots$ (c) $-7, -8.5, -10, -11.5, \ldots$

(2) Find the n^{th} term in the following arithmetic sequences:
(a) $44, 46, 48, \ldots$ (b) $-3, -7, -11, -15, \ldots$

(3) If an arithmetic sequence has $u_{20} = 100$, and $u_{22} = 108$, what is the first term?

(4) Use the formula for an arithmetic series to calculate the sum of the first 8 terms of the arithmetic sequence with first term 1 and common difference 10.

(5) (a) Find the values of a and d for the arithmetic sequence: $21, 19, 17, 15, 13, \ldots$
(b) Use the formula for an arithmetic series to calculate $21 + 19 + 17 + 15 + 13$.
(c) Now use the formula to calculate the sum of $21+19+17+\ldots+1$.

(6) Is the sequence $u_n = -4n + 2$ arithmetic? If so, what is the common difference?

2.4. Geometric Sequences

A *geometric sequence* is one in which each term can be obtained by multiplying the previous term by a fixed number, called the *common ratio*.

EXAMPLES 2.3: *Geometric Sequences*

(i) $\frac{1}{2}, 1, 2, 4, 8, \ldots$ Each term is double the previous one. The common ratio is 2.

(ii) $81, 27, 9, 3, 1, \ldots$ The common ratio is $\frac{1}{3}$.

In a geometric sequence with first term a and common ratio r, the formula for the n^{th} term is:
$u_n = ar^{n-1}$

EXAMPLES 2.4: Consider the geometric sequence with first term 2 and common ratio 1.1.

(i) What is the 10^{th} term?
Applying the formula, with $a = 2$ and $r = 1.1$, $u_{10} = 2 \times (1.1)^9 = 4.7159$

(ii) Which terms of the sequence are greater than 20?
The n^{th} term is given by $u_n = 2 \times (1.1)^{n-1}$. It exceeds 20 if:

$$2 \times (1.1)^{n-1} > 20$$
$$(1.1)^{n-1} > 10$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} \log_{10}(1.1)^{n-1} &> \log_{10} 10 \\ (n-1)\log_{10} 1.1 &> 1 \\ n &> \frac{1}{\log_{10} 1.1} + 1 = 25.2 \end{aligned}$$

So all terms from the 26^{th} onwards are greater than 20.

2.5. Geometric Series

Suppose we want to find the sum of the first 10 terms of the geometric sequence with first term 3 and common ratio 0.5:

$$S_{10} = 3 + 1.5 + 0.75 + \cdots + 3 \times (0.5)^9$$

There is a general formula:

> For a geometric sequence with first term a and common ratio r, the sum of the first n terms is:
> $$S_n = \frac{a(1-r^n)}{1-r}$$

So the answer is: $$S_{50} = \frac{3(1-(0.5)^{10})}{1-0.5} = 5.994$$

2.6. To Prove the Formula for a Geometric Series

Write down the series and then multiply it by r:

$$\begin{array}{rccccccccccc} S_n = & a & + & ar & + & ar^2 & + & ar^3 & + & \dots & + & ar^{n-1} & \\ rS_n = & & & ar & + & ar^2 & + & ar^3 & + & \dots & + & ar^{n-1} & + ar^n \end{array}$$

Subtract the second equation from the first:

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ \implies S_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

Exercises 3.3: **Geometric Sequences and Series**

(1) Find the 8^{th} term and the n^{th} term in the geometric sequence: $5, 10, 20, 40, \dots$

(2) Find the 15^{th} term and the n^{th} term in the geometric sequence: $-2, 4, -8, 16, \dots$

(3) In the sequence $1, 3, 9, 27, \dots$, which is the first term term greater than 1000?

(4) (a) Using the notation above, what are the values of a and r for the sequence: $4, 2, 1, 0.5, 0.25, \dots$?

(b) Use the formula for a geometric series to calculate: $4 + 2 + 1 + 0.5 + 0.25$.

(5) Find the sum of the first 10 terms of the geometric sequence: $4, 16, 64, \dots$

(6) Find the sum of the first n terms of the geometric sequence: $20, 4, 0.8, \dots$ Simplify your answer as much as possible.

(7) Use the formula for a geometric series to show that: $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} = \frac{16 - x^4}{16 - 8x}$

Further reading and exercises

- For more practice with arithmetic and geometric sequences and series, you could use an A-level pure maths textbook.
- *Jacques* §3.3 Geometric Series.

3. Economic Application: Interest Rates, Savings and Loans

Suppose that you invest £500 at the bank, at a fixed interest rate of 6% (that is, 0.06) per annum, and the interest is paid at the end of each year. At the end of one year you receive an interest payment of $0.06 \times 500 = £30$, which is added to your account, so you have £530. After two years, you receive an interest payment of $0.06 \times 530 = £31.80$, so that you have £561.80 in total, and so on.[1]

More generally, if you invest an amount P (the "principal") and interest is paid annually at interest rate i, then after one year you have a total amount y_1:

$$y_1 = P(1+i)$$

after two years:

$$y_2 = (P(1+i))(1+i) = P(1+i)^2$$

and after t years:

$$y_t = P(1+i)^t$$

This is a geometric sequence with common ratio $(1+i)$.

EXAMPLES 3.1: If you save £500 at a fixed interest rate of 6% paid annually:

(i) How much will you have after 10 years?
Using the formula above, $y_{10} = 500 \times 1.06^{10} = £895.42$.

(ii) How long will you have to wait to double your initial investment?
The initial amount will have doubled when:

$$\begin{aligned} 500 \times (1.06)^t &= 1000 \\ \implies (1.06)^t &= 2 \end{aligned}$$

Taking logs of both sides (see chapter 1, section 5):

$$\begin{aligned} t \log_{10} 1.06 &= \log_{10} 2 \\ t &= \frac{\log_{10} 2}{\log_{10} 1.06} = 11.8957 \end{aligned}$$

So you will have to wait 12 years.

3.1. Interval of Compounding

In the previous section we assumed that interest was paid annually. However, in practice, financial institutions often pay interest more frequently, perhaps quarterly or even monthly. We call the time period between interest payments the interval of compounding.

Suppose the bank has a *nominal* (that is, stated) interest rate i, but pays interest m times a year at a rate of $\frac{i}{m}$. After 1 year you would have:

$$P\left(1+\frac{i}{m}\right)^m$$

and after t years:

$$P\left(1+\frac{i}{m}\right)^{mt}$$

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[1] If you are not confident with calculations involving percentages, work through *Jacques* Chapter 3.1

EXAMPLES 3.2: You invest £1000 for two years in the bank, which pays interest at a nominal rate of 8%.

(i) How much will you have at the end of one year if the bank pays interest annually?
You will have: $1000 \times 1.08 = £1080$.

(ii) How much will you have at the end of one year if the bank pays interest quarterly?
Using the formula above with $m = 4$, you will have $1000 \times 1.02^4 = £1082.43$.
Note that you are better off (for a given nominal rate) if the interval of compounding is shorter.

(iii) How much will you have at the end of 5 years if the bank pays interest monthly?
Using the formula with $m = 12$ and $t = 5$, you will have:

$$1000 \times \left(1 + \frac{0.08}{12}\right)^{5\times 12} = £1489.85$$

From this example, you can see that if the bank pays interest quarterly and the nominal rate is 8%, then your investment actually grows by 8.243% in one year. The *effective* annual interest rate is 8.243%. In the UK this rate is known as the *Annual Equivalent Rate (AER)* (or sometimes the *Annual Percentage Rate (APR)*). Banks often describe their savings accounts in terms of the AER, so that customers do not need to do calculations involving the interval of compounding.

If the nominal interest rate is i, and interest is paid m times a year, an investment P grows to $P\,(1 + i/m)^m$ in one year. So the formula for the Annual Equivalent Rate is:

$$AER = \left(1 + \frac{i}{m}\right)^m - 1$$

EXAMPLES 3.3: *Annual Equivalent Rate*
If the nominal interest rate is 6% and the bank pays interest monthly, what is the AER?

$$AER = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.0617$$

The Annual Equivalent Rate is 6.17%.

3.2. Regular Savings

Suppose that you invest an amount A at the beginning of every year, at a fixed interest rate i (compounded annually). At the end of t years, the amount you invested at the beginning of the first year will be worth $A(1+i)^t$, the amount you invested in the second year will be worth $A(1+i)^{t-1}$, and so on. The total amount that you will have at the end of t years is:

$$\begin{aligned} S_t &= A(1+i)^t + A(1+i)^{t-1} + A(1+i)^{t-2} + \cdots + A(1+i)^2 + A(1+i) \\ &= A(1+i) + A(1+i)^2 + A(1+i)^3 + \cdots + A(1+i)^{t-1} + A(1+i)^t \end{aligned}$$

This is the sum of the first t terms of a geometric sequence with first term $A(1+i)$, and common ratio $(1+i)$. We can use the formula from section 2.5. The sum is:

$$S_t = \frac{A(1+i)\left(1-(1+i)^t\right)}{1-(1+i)} = \frac{A(1+i)}{i}\left((1+i)^t - 1\right)$$

So, for example, if you saved £200 at the beginning of each year for 10 years, at 5% interest, then you would accumulate $200\frac{1.05}{0.05}((1.05)^{10} - 1) = £2641.36$.

3.3. Paying Back a Loan

If you borrow an amount L, to be paid back in annual repayments over t years, and the interest rate is i, how much do you need to repay each year?

Let the annual repayment be y. At the end of the first year, interest will have been added to the loan. After repaying y you will owe:

$$X_1 = L(1+i) - y$$

At the end of two years you will owe:

$$\begin{aligned} X_2 &= (L(1+i) - y)(1+i) - y \\ &= L(1+i)^2 - y(1+i) - y \end{aligned}$$

At the end of three years: $X_3 = L(1+i)^3 - y(1+i)^2 - y(1+i) - y$

and at the end of t years: $X_t = L(1+i)^t - y(1+i)^{t-1} - y(1+i)^{t-2} - \ldots - y$

But if you are to pay off the loan in t years, X_t must be zero:

$$\implies L(1+i)^t = y(1+i)^{t-1} + y(1+i)^{t-2} + \ldots + y$$

The right-hand side of this equation is the sum of t terms of a geometric sequence with first term y and common ratio $(1+i)$ (in reverse order). Using the formula from section 2.5:

$$\begin{aligned} L(1+i)^t &= \frac{y\big((1+i)^t - 1\big)}{i} \\ \implies y &= \frac{Li(1+i)^t}{((1+i)^t - 1)} \end{aligned}$$

This is the amount that you need to repay each year.

EXERCISES 3.4: **Interest Rates, Savings, and Loans**
(Assume annual compounding unless otherwise specified.)

(1) Suppose that you save £300 at a fixed interest rate of 4% per annum.
   (a) How much would you have after 4 years if interest were paid annually?
   (b) How much would you have after 10 years if interest were compounded monthly?
(2) If you invest £20 at 15% interest, how long will it be before you have £100?
(3) If a bank pays interest daily and the nominal rate is 5%, what is the AER?
(4) If you save £10 at the beginning of each year for 20 years, at an interest rate of 9%, how much will you have at the end of 20 years?
(5) Suppose you take out a loan of £100000, to be repaid in regular annual repayments, and the annual interest rate is 5%.
   (a) What should the repayments be if the loan is to be repaid in 25 years?
   (b) Find a formula for the repayments if the repayment period is T years.

Further reading and exercises

- *Jacques* §3.1 and 3.2.
- *Anthony & Biggs* Chapter 4

4. Present Value and Investment

Would you prefer to receive (a) a gift of £1000 today, or (b) a gift of £1050 in one year's time?

Your decision (assuming you do not have a desperate need for some immediate cash) will depend on the interest rate. If you accepted the £1000 today, and saved it at interest rate i, you would have £$1000(1+i)$ in a year's time. We could say:

$$\begin{aligned} \text{Future value of (a)} &= 1000(1+i) \\ \text{Future value of (b)} &= 1050 \end{aligned}$$

You should accept the gift that has higher future value. For example, if the interest rate is 8%, the future value of (a) is £1080, so you should accept that. But if the interest rate is less than 5%, it would be better to take (b).

Another way of looking at this is to consider what cash sum *now* would be equivalent to a gift of a gift of £1050 in one year's time. An amount P received now would be equivalent to an amount £1050 in one year's time if:

$$\begin{aligned} P(1+i) &= 1050 \\ \implies P &= \frac{1050}{1+i} \end{aligned}$$

We say that:

The *Present Value* of "£1050 in one year's time" is $\dfrac{1050}{1+i}$

More generally:

> If the annual interest rate is i, the *Present Value* of an amount A to be received in t years' time is:
> $$P = \frac{A}{(1+i)^t}$$

The present value is also known as the *Present Discounted Value*; payments received in the future are worth less – we "discount" them at the interest rate i.

EXAMPLES 4.1: *Present Value and Investment*

(i) The prize in a lottery is £5000, but the prize will be paid in two years' time. A friend of yours has the winning ticket. How much would you be prepared to pay to buy the ticket, if you are able to borrow and save at an interest rate of 5%?

The present value of the ticket is:

$$P = \frac{5000}{(1.05)^2} = 4535.15$$

This is the maximum amount you should pay. If you have £4535.15, you would be indifferent between (a) paying this for the ticket, and (b) saving your money at 5%. Or, if you don't have any money at the moment, you would be indifferent between (a) taking out a loan of £4535.15, buying the ticket, and repaying the loan after 2 years when you receive the prize, and (b) doing nothing. Either way, if your friend will sell the ticket for less than £4535.15, you should buy it.

We can see from this example that *Present Value* is a powerful concept: a single calculation of the PV enables you to answer the question, without thinking about exactly how the money to buy the ticket is to be obtained. This does rely, however, on the assumption that you can borrow and save at the same interest rate.

(ii) An investment opportunity promises you a payment of £1000 at the end of each of the next 10 years, and a capital sum of £5000 at the end of the 11th year, for an initial outlay of £10000. If the interest rate is 4%, should you take it?

We can calculate the present value of the investment opportunity by adding up the present values of all the amounts paid out and received:

$$P = -10000 + \frac{1000}{1.04} + \frac{1000}{1.04^2} + \frac{1000}{1.04^3} + \cdots + \frac{1000}{1.04^{10}} + \frac{5000}{1.04^{11}}$$

In the middle of this expression we have (again) a geometric series. The first term is $\frac{1000}{1.04}$ and the common ratio is $\frac{1}{1.04}$. Using the formula from section 2.5:

$$\begin{aligned} P &= -10000 + \frac{\frac{1000}{1.04}\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{1 - \frac{1}{1.04}} + \frac{5000}{1.04^{11}} \\ &= -10000 + \frac{1000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right)}{0.04} + 3247.90 \\ &= -10000 + 25000\left(1 - \left(\frac{1}{1.04}\right)^{10}\right) + 3247.90 \\ &= -10000 + 8110.90 + 3247.90 = -10000 + 11358.80 = £1358.80 \end{aligned}$$

The present value of the opportunity is positive (or equivalently, the present value of the return is greater than the initial outlay): you should take it.

4.1. Annuities

An *annuity* is a financial asset which pays you an amount A each year for N years. Using the formula for a geometric series, we can calculate the present value of an annuity:

$$\begin{aligned} PV &= \frac{A}{1+i} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \cdots + \frac{A}{(1+i)^N} \\ &= \frac{\frac{A}{1+i}\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{1 - \left(\frac{1}{1+i}\right)} \\ &= \frac{A\left(1 - \left(\frac{1}{1+i}\right)^N\right)}{i} \end{aligned}$$

The present value tells you the price you would be prepared to pay for the asset.

Exercises 3.5: **Present Value and Investment**

(1) On your 18^{th} birthday, your parents promise you a gift of £500 when you are 21. What is the present value of the gift (a) if the interest rate is 3% (b) if the interest rate is 10%?

(2) (a) How much would you pay for an annuity that pays £20 a year for 10 years, if the interest rate is 5%?

(b) You buy it, then after receiving the third payment, you consider selling the annuity. What price will you be prepared to accept?

(3) The useful life of a bus is five years. Operating the bus brings annual profits of £10000. What is the value of a new bus if the interest rate is 6%?

(4) An investment project requires an initial outlay of £2400, and can generate revenue of £2000 per year. In the first year, operating costs are £600; thereafter operating costs increase by £500 a year.

(a) What is the maximum length of time for which the project should operate?

(b) Should it be undertaken if the interest rate is 5%?

(c) Should it be undertaken if the interest rate is 10%?

Further reading and exercises

- *Jacques* §3.4
- *Anthony & Biggs* Chapter 4
- *Varian* also discusses Present Value and has more economic examples.

5. Limits

5.1. The Limit of a Sequence

If we write down some of the terms of the geometric sequence: $u_n = (\frac{1}{2})^n$:

$$\begin{aligned} u_1 &= (\tfrac{1}{2})^1 &= 0.5 \\ u_{10} &= (\tfrac{1}{2})^{10} &= 0.000977 \\ u_{20} &= (\tfrac{1}{2})^{20} &= 0.000000954 \end{aligned}$$

we can see that as n gets larger, u_n gets closer and closer to zero. We say that "the limit of the sequence as n tends to infinity is zero" or "the sequence converges to zero" or:

$$\lim_{n\to\infty} u_n = 0$$

EXAMPLES 5.1: *Limits of Sequences*

(i) $u_n = 4 - (0.1)^n$
The sequence is:

$$3.9, 3.99, 3.999, 3.9999, \ldots$$

We can see that it converges:

$$\lim_{n\to\infty} u_n = 4$$

(ii) $u_n = (-1)^n$
This sequence is $-1, +1, -1, +1, -1, +1, \ldots$ It has no limit.

(iii) $u_n = \frac{1}{n}$
The terms of this sequence get smaller and smaller:

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots$$

It converges to zero:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

(iv) $2, 4, 8, 16, 32, \ldots$
This is a geometric sequence with common ratio 2. The terms get bigger and bigger. It *diverges*:

$$u_n \to \infty \quad \text{as } n \to \infty$$

(v) $u_n = \dfrac{2n^3 + n^2}{3n^3}$.

A useful trick is to divide the numerator and the denominator by the highest power of n; that is, by n^3. Then:

$$u_n = \left(\frac{2 + \frac{1}{n}}{3}\right)$$

and we know that $\frac{1}{n} \to 0$, so:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{2 + \frac{1}{n}}{3}\right) = \frac{2}{3}$$

EXERCISES 3.6: Say whether each if the following sequences converges or diverges as $n \to \infty$. If it converges, find the limit.

(1) $u_n = (\frac{1}{3})^n$
(2) $u_n = -5 + (\frac{1}{4})^n$
(3) $u_n = (-\frac{1}{3})^n$
(4) $u_n = 7 - (\frac{2}{5})^n$
(5) $u_n = \frac{10}{n^3}$
(6) $u_n = (1.2)^n$
(7) $u_n = 25n + \frac{10}{n^3}$
(8) $u_n = \frac{7n^2+5n}{n^2}$

From examples like these we can deduce some general results that are worth remembering:

- $\lim_{n\to\infty} \frac{1}{n} = 0$
- Similarly $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^3} = 0$ etc
- If $|r| < 1$, $\lim_{n\to\infty} r^n = 0$
- If $r > 1$, $r^n \to \infty$

5.2. Infinite Geometric Series

Consider the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ It is a geometric sequence with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We can find the sum of the first n terms:

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}$$

using the formula from section 2.5:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\tfrac{1}{2}\right)^n\right) \\ &= 2 - \left(\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

As the number of terms gets larger and larger, their sum gets closer and closer to 2:

$$\lim_{n\to\infty} S_n = 2$$

Equivalently, we can write this as:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$$

So we have found the sum of an *infinite* number of terms to be a finite number. Using the sigma notation this can be written:

$$\sum_{i=1}^{\infty} \left(\tfrac{1}{2}\right)^{i-1} = 2$$

or (a little more neatly):

$$\sum_{i=0}^{\infty}\left(\tfrac{1}{2}\right)^i = 2$$

The same procedure works for any geometric series with common ratio r, provided that $|r| < 1$. The sum of the first n terms is:

$$S_n = \frac{a(1-r^n)}{1-r}$$

As $n \to \infty$, $r^n \to 0$ so the series converges:

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

or equivalently:

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r}$$

But note that if $|r| > 1$ the terms of the series get bigger and bigger, so it *diverges*: the infinite sum sum does not exist.

EXAMPLES 5.2: Find the sum to infinity of the following series:

(i) $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} \ldots$
This is a geometric series, with $a = 2$ and $r = -\frac{1}{2}$. It converges because $|r| < 1$. Using the formula above:

$$S_\infty = \frac{a}{1-r} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

(ii) $x + 2x^2 + 4x^3 + 8x^4 + \ldots$ (assuming $0 < x < 0.5$)
This is a geometric series with $a = x$ and $r = 2x$. We know $0 < r < 1$, so it converges. The formula for the sum to infinity gives:

$$S_\infty = \frac{a}{1-r} = \frac{x}{1-2x}$$

5.3. Economic Application: Perpetuities

In section 4.1 we calculated the present value of an annuity – an asset that pays you an amount A each year for a fixed number of years. A *perpetuity* is an asset that pays you an amount A each year *forever*.
If the interest rate is i, the present value of a perpetuity is:

$$PV = \frac{A}{1+i} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots$$

This is an infinite geometric series. The common ratio is $\frac{1}{1+i}$. Using the formula for the sum of an infinite series:

$$\begin{aligned} PV &= \frac{\frac{A}{1+i}}{1-\left(\frac{1}{1+i}\right)} \\ &= \frac{A}{i} \end{aligned}$$

Again, the present value tells you the price you would be prepared to pay for the asset. Even an asset that pays out forever has a finite price. (Another way to get this result is to let

$N \to \infty$ in the formula for the present value of an annuity that we obtained earlier.)

EXERCISES 3.7: **Infinite Series, and Perpetuities**

(1) Evaluate the following infinite sums:

(a) $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \ldots$

(b) $1 + 0.2 + 0.04 + 0.008 + 0.0016 + \ldots$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$

(d) $\frac{2}{3} + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^7 + \ldots$

(e) $\sum_{r=3}^{\infty} \left(\frac{1}{2}\right)^r$

(f) $\sum_{r=0}^{\infty} x^r$ assuming $|x| < 1$. (Why is this assumption necessary?)

(g) $\frac{2y^2}{x} + \frac{4y^3}{x^2} + \frac{8y^4}{x^3} + \ldots$ What assumption is needed here?

(2) If the interest rate is 4%, what is the present value of:

(a) an annuity that pays £100 each year for 20 years?

(b) a perpetuity that pays £100 each year forever?

How will the value of each asset have changed after 10 years?

(3) A firm's profits are expected to be £1000 this year, and then to rise by 2% each year after that (forever). If the interest rate is 5%, what is the present value of the firm?

Further reading and exercises

- *Anthony & Biggs*: §3.3 discusses limits briefly.
- *Varian* has more on financial assets including perpetuities, and works out the present value of a perpetuity in a different way.
- For more on limits of sequences, and infinite sums, refer to an A-level pure maths textbook.

6. The Number e

If we evaluate the numbers in the sequence:

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

we get: $u_1 = 2$, $u_2 = (1 + \frac{1}{2})^2 = 2.25$, $u_3 = (1 + \frac{1}{3})^3 = 2.370, \ldots$
For some higher values of n we have, for example:

$$\begin{aligned} u_{10} &= (1.1)^{10} &&= 2.594 \\ u_{100} &= (1.01)^{100} &&= 2.705 \\ u_{1000} &= (1.001)^{1000} &&= 2.717 \\ u_{10000} &= (1.0001)^{10000} &&= 2.71814 \\ u_{100000} &= (1.00001)^{100000} &&= 2.71826 \quad \ldots \end{aligned}$$

As $n \to \infty$, u_n gets closer and closer to a limit of $2.718281828459\ldots$ This is an irrational number (see Chapter 1) known simply as e. So:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \quad (\approx 2.71828)$$

e is important in calculus (as we will see later) and arises in many economic applications.
We can generalise this result to:

> For any value of r, $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$

Exercises 3.8: Verify (approximately, using a calculator) that $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2$.
Hint: Most calculators have a button that evaluates e^x for any number x.

6.1. Economic Application: Continuous Compounding

Remember, from section 3.1, that if interest is paid m times a year and the nominal rate is i, then the return after t years from investing an inital amount P is:

$$P\left(1 + \frac{i}{m}\right)^{mt}$$

Interest might be paid quarterly ($m = 4$), monthly ($m = 12$), weekly ($m = 52$), or daily ($m = 365$). Or it could be paid even more frequently – every hour, every second ... As the interval of compounding get shorter, interest is compounded almost continuously.
As $m \to \infty$, we can apply our result above to say that:

$$\lim_{m\to\infty} \left(1 + \frac{i}{m}\right)^m = e^i$$

and so:

> If interest is compounded continuously at rate i, the return after t years on an initial amount P is:
> Pe^{it}

Examples 6.1: If interest is compounded continuously, what is the AER if:

(i) the interest rate is 5%?
Applying the formula, an amount P invested for one year yields:

$$Pe^{0.05} = 1.05127P = (1 + 0.05127)P$$

So the AER is 5.127%.

(ii) the interest rate is 8%?
Similarly, $e^{0.08} = 1.08329$, so the AER is 8.329%.

We can see from these examples that with continuous compounding the AER is little different from the interest rate. So, when solving economic problems we often simplify by assuming continuous compounding, because it avoids the messy calculations for the interval of compounding.

6.2. Present Value with Continuous Compounding

In section 4, when we showed that the present value of an amount A received in t years time is $\frac{A}{(1+i)^t}$, we were assuming annual compounding of interest.

> With continuous compounding, if the interest rate is i, the present value of an amount A received in t years is:
> $P = Ae^{-it}$

Continuous compounding is particularly useful because it allows us to calculate the present value when t is not a whole number of years.

To see where the formula comes from, note that if you have an amount Ae^{-it} now, and you save it for t years with continuous compounding, you will then have $Ae^{-it}e^{it} = A$. So "Ae^{-it} now" and "A after t years", are worth the same.

Exercises 3.9: e

(1) Express the following in terms of e:

(a) $\lim_{n\to\infty}(1+\frac{1}{n})^n$ (b) $\lim_{n\to\infty}(1+\frac{5}{n})^n$ (c) $\lim_{n\to\infty}(1+\frac{1}{2n})^n$

(2) If you invest £100, the interest rate is 5%, and interest is compounded countinuously:

(a) How much will you have after 1 year?

(b) How much will you have after 5 years?

(c) What is the AER?

(3) You expect to receive a gift of £100 on your next birthday. If the interest rate is 5%, what is the present value of the gift (a) six months before your birthday (b) 2 days before your birthday?

Further reading and exercises

- *Anthony & Biggs*: §7.2 and §7.3.
- *Jacques* §2.4.

Solutions to Exercises in Chapter 3

Exercises 3.1:

(1) (a) 14
(b) 2.75
(c) -13
(d) 2.25
(e) -6

(2) $-6, 6, 26$

(3) $4, 4, 4, 4, 4$

(4) $u_4 = -2$

(5) 30

(6) (a) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$
(b) $1 + 2 + 4 + 8$
(c) $1+3+5+\ldots+(2n+1)$

(7) (a) n
(b) $4n - 3$

(8) There are several possibilities. e.g.
(a) $\sum_{r=1}^{25} r^2$
(b) $\sum_{r=2}^{7} 3r$
(c) $\sum_{r=4}^{n} r^2$

Exercises 3.2:

(1) (a) $a = 4$ $d = 3$
(b) $a = -1$ $d = 3$
(c) $a = -7$ $d = -1.5$

(2) (a) $u_n = 44 + 2(n-1)$ $= 42 + 2n$
(b) $u_n = -3 - 4(n-1)$ $= -4n + 1$

(3) $u_1 = 24$

(4) 288

(5) (a) $a = 21$ $d = -2$
(b) $S_5 = 85$
(c) $S_{11} = 121$

(6) Yes, $u_n = -2 - 4(n-1)$, so $d = -4$

Exercises 3.3:

(1) 540, $5 \times 2^{n-1}$

(2) -32768, $(-2)^n$

(3) 8^{th}

(4) (a) $a = 4$, $r = \frac{1}{2}$
(b) $S_5 = 7.75$

(5) $S_{10} = 1398100$

(6) $S_n = 25(1 - \frac{1}{5}^n)$
$= 25 - \left(\frac{1}{5}\right)^{n-2}$

(7) $a = 1; r = \left(\frac{x}{2}\right); n = 4$
$\Rightarrow S_4 = \frac{16-x^4}{16-8x}$

Exercises 3.4:

(1) (a) 350.96
(b) 447.25

(2) $t = 12$

(3) 5.13%

(4) $S_{20} = £557.65$

(5) (a) £8024.26
(b) $y = \frac{5000(1.05)^T}{1.05^T - 1}$

Exercises 3.5:

(1) (a) £457.57
(b) £375.66

(2) (a) £154.43
(b) £115.73

(3) £42123

(4) (a) 3 years.
(b) Yes PV=£95.19
(c) No PV=$-$£82.95

Exercises 3.6:

(1) $u_n \to 0$ as $n \to \infty$

(2) $u_n \to -5$ as $n \to \infty$

(3) $u_n \to 0$ as $n \to \infty$

(4) $u_n \to 7$ as $n \to \infty$

(5) $u_n \to 0$ as $n \to \infty$

(6) $u_n \to \infty$ as $n \to \infty$

(7) $u_n \to \infty$ as $n \to \infty$

(8) $u_n \to 7$ as $n \to \infty$

Exercises 3.7:

(1) (a) $\frac{1}{2}$
(b) $\frac{5}{4}$
(c) $\frac{2}{3}$
(d) $\frac{18}{19}$
(e) $\frac{1}{4}$
(f) $\frac{1}{1-x}$ If $|x| \geq 1$ sequence diverges
(g) $\frac{2y^2}{x-2y}$ assuming $|2y| < |x|$

(2) (a) £1359.03. Is only worth £811.09 after ten years.
(b) £2500. Same value in ten years.

(3) £33,333.33

Exercises 3.8:

(1) ($e^2 = 7.389$ to 3 decimal places)

Exercises 3.9:

(1) (a) e
(b) e^5
(c) $e^{\frac{1}{2}}$

(2) (a) £105.13
(b) £128.40
(c) 5.13%

(3) (a) $P = 100e^{-\frac{i}{2}}$ $= £97.53$
(b) $P = 100e^{-\frac{2i}{365}}$ $= £99.97$

[2] This Version of Workbook Chapter 3: September 25, 2006

Worksheet 3: Sequences, Series, and Limits; the Economics of Finance

Quick Questions

(1) What is the n^{th} term of each of the following sequences:
(a) $20, 15, 10, 5, \ldots$ (b) $1, 8, 27, 64, \ldots$ (c) $0.2, 0.8, 3.2, 12.8, \ldots$

(2) Write out the series: $\sum_{r=0}^{n-1}(2r-1)^2$ without using sigma notation, showing the first four terms and the last two terms.

(3) For each of the following series, work out how many terms there are and hence find the sum:
(a) $3+4+5+\cdots+20$ (b) $1+0.5+0.25+\cdots+(0.5)^{n-1}$ (c) $5+10+20+\cdots+5\times 2^n$

(4) Express the series $3+7+11+\cdots+(4n-1)+(4n+3)$ using sigma notation.

(5) If you invest £500 at a fixed interest rate of 3% per annum, how much will you have after 4 years:
(a) if interest is paid annually?
(b) if interest is paid monthly? What is the AER in this case?
(c) if interest is compounded continuously?
If interest is paid annually, when will your savings exceed £600?

(6) If the interest rate is 5% per annum, what is the present value of:
(a) An annuity that pays £100 a year for 20 years?
(b) A perpetuity that pays £50 a year?

(7) Find the limit, as $n \to \infty$, of:
(a) $3(1+(0.2)^n)$ (b) $\dfrac{5n^2+4n+3}{2n^2+1}$ (c) $0.75+0.5625+\cdots+(0.75)^n$

Longer Questions

(1) Carol (an economics student) is considering two possible careers. As an acrobat, she will earn £30000 in the first year, and can expect her earnings to increase at 1% per annum thereafter. As a beekeeper, she will earn only £20000 in the first year, but the subsequent increase will be 5% per annum. She plans to work for 40 years.
(a) If she decides to be an acrobat:
(i) How much will she earn in the 3^{rd} year of her career?
(ii) How much will she earn in the n^{th} year?
(iii) What will be her total career earnings?
(b) If she decides to be a beekeeper:
(i) What will be her total career earnings?
(ii) In which year will her annual earnings first exceed what she would have earned as an acrobat?
(c) She knows that what matters for her choice of career is the present value of her earnings. The rate of interest is i. (Assume that earnings are received at the end of each year, and that her choice is made on graduation day.) If she decides to be an acrobat:

   (i) What is the present value of her first year's earnings?
   (ii) What is the present value of the n^{th} year's earnings?
   (iii) What is the total present value of her career earnings?
(d) Which career should she choose if the interest rate is 3%?
(e) Which career should she choose if the interest rate is 15%?
(f) Explain these results.

(2) Bill is due to start a four year degree course financed by his employer and he feels the need to have his own computer. The computer he wants cost £1000. The insurance premium on the computer will start at £40 for the first year, and decline by £5 per year throughout the life of the computer, while repair bills start at £50 in the computer's first year, and increase by 50% per annum thereafter. (The grant and the insurance premium are paid at the beginning of each year, and repair bills at the end.)
The resale value for computers is given by the following table:

	Resale value at end of year
Year 1	75% of initial cost
Year 2	60% of initial cost
Year 3	20% of initial cost
Year 4 and onwards	£100

The interest rate is 10% per annum.

(a) Bill offers to forgo £360 per annum from his grant if his employer purchases a computer for him and meets all insurance and repair bills. The computer will be sold at the end of the degree course and the proceeds paid to the employer. Should the employer agree to the scheme? If not, what value of computer would the employer agree to purchase for Bill (assuming the insurance, repair and resale schedules remain unchanged)?

(b) Bill has another idea. He still wants the £1000 computer, but suggests that it be replaced after two years with a new one. Again, the employer would meet all bills and receive the proceeds from the sale of both computers, and Bill would forgo £360 per annum from his grant. How should the employer respond in this case?