CHAPTER 4

Functions

> Functions, and the language of functions, are widely used in economics. Linear and quadratic functions were discussed in Chapter 2. Now we introduce other useful functions (including **exponential** and **logarithmic** functions) and concepts (such as **inverse functions** and **functions of several variables**). Economic applications are **supply and demand functions**, **utility functions** and **production functions**.

1. Function Notation, and Some Common Functions

We have already encountered some functions in Chapter 2. For example:

$$y = 5x - 8$$

Here y is a function of x (or in other words, y depends on x).

$$C = 3 + 2q^2$$

Here, a firm's total cost, C, of producing output is a function of the quantity of output, q, that it produces.

To emphasize that *y is a function of x*, and that *C is a function of q*, we often write:

$$y(x) = 5x - 8 \quad \text{and} \quad C(q) = 3 + 2q^2$$

Also, for general functions it is common to use the letter f rather than y:

$$f(x) = 5x - 8$$

Here, x is called the *argument* of the function.

In general, we can think of a function $f(x)$ as a "black box" which takes x as an input, and produces an output $f(x)$:

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x)$$

For each input value, there is a unique output value.

Examples 1.1: For the function $f(x) = 5x - 8$

(i) Evaluate $f(3)$

$$f(3) = 15 - 8 = 7$$

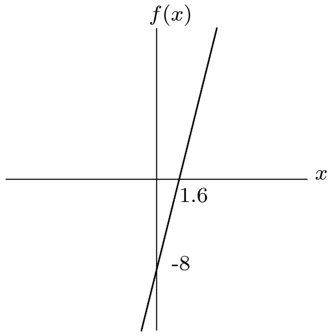
(ii) Evaluate $f(0)$

$$f(0) = 0 - 8 = -8$$

(iii) Solve the equation $f(x) = 0$

$$\begin{aligned} f(x) &= 0 \\ \implies 5x - 8 &= 0 \\ \implies x &= 1.6 \end{aligned}$$

(iv) Hence sketch the graph.



> Exercises 4.1: **Using Function Notation**
> (1) For the function $f(x) = 9 - 2x$, evaluate $f(2)$ and $f(-4)$.
> (2) Solve the equation $g(x) = 0$, where $g(x) = 5 - \frac{10}{x+1}$.
> (3) If a firm has cost function $C(q) = q^3 - 5q$, what are its costs of producing 4 units of output?

1.1. Polynomials

Polynomials were introduced in Chapter 1. They are functions of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n+1} + ... + a_0$$

Examples 1.2:

(i) $g(x) = 5x^3 - 2x + 1$ is a polynomial of degree 3

(ii) $h(x) = 5 + x^4 - x^9 + x^2$ is a polynomial of degree 9

A linear function is a polynomial of degree 1, and a quadratic function is a polynomial of degree 2. We already know what their graphs look like (see Chapter 2); a linear function crosses the x-axis once (or not at all); a quadratic function crosses the x-axis twice, or touches it once, or not at all.

> The graph of a polynomial of degree n crosses the x-axis up to n times.

EXERCISES 4.2: **Polynomials**

(1) For the polynomial function $f(x) = x^3 - 3x^2 + 2x$:

(a) Factorise the function and hence show that it crosses the x-axis at $x = 0$, $x = 1$ and $x = 2$.

(b) Check whether the function is positive or negative when $x < 0$, when $0 < x < 1$, when $1 < x < 2$, and when $x > 2$ and hence sketch a graph of the function.

(2) Consider the polynomial function $g(x) = 5x^2 - x^4 - 4$.

(a) What is the degree of the polynomial?

(b) Factorise the function. (*Hint*: it is a quadratic in x^2: $-(x^2)^2 + 5x^2 - 4$)

(c) Hence sketch the graph, using the same method as in the previous example.

1.2. The Function $f(x) = x^n$

When n is a positive integer, $f(x) = x^n$ is just a simple polynomial. But n could be negative:

$$f(x) = \frac{1}{x^2} \quad (n = -2)$$

and in economic applications, we often use fractional values of n, which we can do if x represents a positive variable (such as “output”, or “employment”):

$$\begin{aligned} f(x) &= x^{0.3} \\ f(x) &= x^{\frac{3}{2}} \\ f(x) &= \sqrt{x} \quad (n = 0.5) \end{aligned}$$

EXERCISES 4.3: **The Function x^n**

On a single diagram, with x-axis from 0 to 4, and vertical axis from 0 to 6.5, plot carefully the graphs of $f(x) = x^n$, for n=1.3, n=1, n=0.7 and $n = -0.3$.

Hint: You cannot evaluate $x^{-0.3}$ when $x = 0$, but try values of x close to zero.

You should find that your graphs have the standard shapes below.

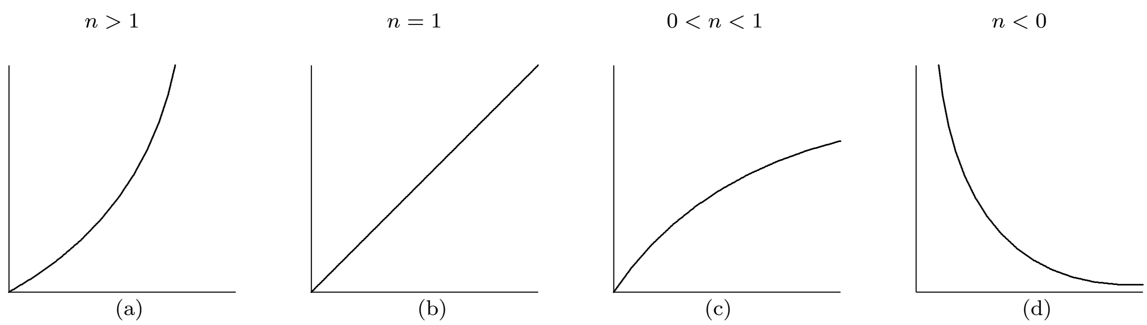


FIGURE 1. The function $f(x) = x^n$

1.3. Increasing and Decreasing Functions

If $f(x)$ increases whenever x increases:

- the graph is upward-sloping;
- we say that f is an *increasing function* of x
  (or sometimes that it is a *monotonic increasing function*).

Similarly, a function that decreases whenever x increases has a downward-sloping graph and is known as a (monotonic) *decreasing function*. *Monotonic* means simply that f moves in one direction (either up or down) as x increases.

EXAMPLES 1.3: $f(x) = x^n \quad (x \geq 0)$
We can see from Figure 1 that this function is monotonic, whatever the value of n.

(i) When $n > 0$ f is an increasing function of x (graphs (a) to (c))

(ii) When $n < 0$ f is an decreasing function of x (graph (d))

1.4. Limits of Functions

In Chapter 3, we looked at limits of sequences. We also use this idea for functions.

EXAMPLES 1.4: $f(x) = x^n$
Looking at Figure 1 again:

(i) If $n > 0$, $\lim_{x\to\infty} x^n = \infty$ and $\lim_{x\to 0} x^n = 0$

(ii) If $n < 0$, $\lim_{x\to\infty} x^n = 0$ and $\lim_{x\to 0} x^n = \infty$

EXERCISES 4.4: **Increasing and decreasing functions, and limits**

(1) If $f(x) = 2 - x^2$ find:

(a) $\lim_{x\to\infty} f(x)$

(b) $\lim_{x\to -\infty} f(x)$

(c) $\lim_{x\to 0} f(x)$

(2) If $y = \frac{2}{x} + 3$ for values of $x \geq 0$, is y an increasing or a decreasing function of x? What is the limit of y as x tends to infinity?

(3) If $g(x) = 1 - \frac{5}{x^2}$ for $x \geq 0$:

(a) Is g an increasing or a decreasing function?

(b) What is $\lim_{x\to\infty} g(x)$?

(c) What is $\lim_{x\to 0} g(x)$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- Both of the above are brief. For more examples, use an A-level pure maths textbook.

2. Composite Functions

If we have two functions, $f(x)$ and $g(x)$, we can take the output of f and input it to g:

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow f(x) \longrightarrow \boxed{\quad g \quad} \longrightarrow g(f(x))$$

We can think of the final output as the output of a new function, $g(f(x))$, called a *composite function* or a "function of a function."

EXAMPLES 2.1: If $f(x) = 2x + 3$ and $g(x) = x^2$:

(i) $g(f(2)) = g(7) = 49$

(ii) $f(g(2)) = f(4) = 11$

(iii) $f(g(-4)) = f(16) = 35$

(iv) $g(f(-4)) = g(-5) = 25$
Note that $f(g(x))$ is not the same as $g(f(x))$!

EXAMPLES 2.2: If $f(x) = 2x + 3$ and $g(x) = x^2$, what are the functions:

(i) $f(g(x))$?

$$f(g(x)) = 2g(x) + 3 = 2x^2 + 3$$

(ii) $g(f(x))$?

$$g(f(x)) = (f(x))^2 = (2x + 3)^2 = 4x^2 + 12x + 9$$

Note that we can check these answers using the previous ones. For example:

$$f(g(x)) = 2x^2 + 3 \implies f(g(2)) = 11$$

EXERCISES 4.5: **Composite Functions**

(1) If $f(x) = 3 - 2x$ and $g(x) = 8x - 1$ evaluate $f(g(2))$ and $g(f(2))$.

(2) If $g(x) = \frac{4}{x+1}$ and $h(x) = x^2 + 1$, find $g(h(1))$ and $h(g(1))$.

(3) If $k(x) = \sqrt[3]{x}$ and $m(x) = x^3$, evaluate $k(m(3))$ and $m(k(3))$. Why are the answers the same in this case?

(4) If $f(x) = x + 1$, and $g(x) = 2x^2$, what are the functions $g(f(x))$, and $f(g(x))$?

(5) If $h(x) = \frac{5}{x+2}$ and $k(x) = \frac{1}{x}$ find the functions $h(k(x))$, and $k(h(x))$.

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §2.3.
- Both of the above are brief. For more examples, use an A-level maths pure textbook.

3. Inverse Functions

Suppose we have a function $f(x)$, and we call the output y. If we can find another function that takes y as an input and produces the original value x as output, it is called the inverse function $f^{-1}(y)$.

$$x \longrightarrow \boxed{\quad f \quad} \longrightarrow \quad y \quad \longrightarrow \boxed{\quad f^{-1} \quad} \longrightarrow x$$

FIGURE 2. f^{-1} is the inverse of f: $f^{-1}(f(x)) = x$

If we can find such a function, and we take its output and input it to the original function, we find also that f is the inverse of f^{-1}:

$$y \longrightarrow \boxed{\quad f^{-1} \quad} \longrightarrow \quad x \quad \longrightarrow \boxed{\quad f \quad} \longrightarrow y$$

FIGURE 3. f is the inverse of f^{-1}: $f(f^{-1}(y)) = y$

EXAMPLES 3.1: What is the inverse of the function

(i) $f(x) = 3x + 1$?

Call the output of the function y:

$$y = 3x + 1$$

This equation tells you how to find y if you know x (that is, it gives y in terms of x). Now rearrange it, to find x in terms of y (that is, make x the subject):

$$\begin{aligned} 3x &= y - 1 \\ x &= \frac{y-1}{3} \end{aligned}$$

So the inverse function is:

$$f^{-1}(y) = \frac{y-1}{3}$$

(ii) $f(x) = \frac{2}{x-1}$?

$$\begin{aligned} y &= \frac{2}{x-1} \\ y(x-1) &= 2 \\ x &= 1 + \frac{2}{y} \end{aligned}$$

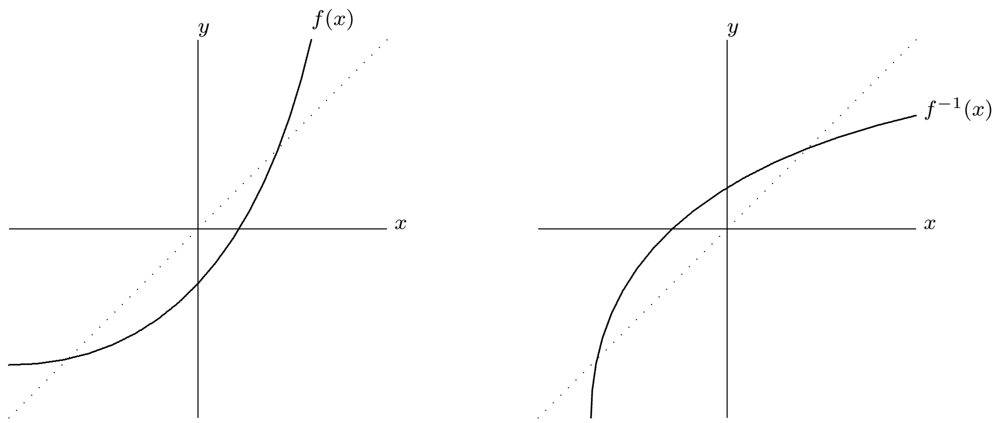
So the inverse function is:

$$f^{-1}(y) = 1 + \frac{2}{y}$$

- It doesn't matter what letter we use for the argument of a function. The function above would still be the same function if we wrote it as:
$$f^{-1}(z) = 1 + \frac{2}{z}$$
- We could say, for example, that
$$f^{-1}(x) = 1 + \frac{2}{x} \text{ is the inverse of } f(x) = \frac{2}{x-1}$$
using the same argument for both (although this can be be confusing).

- *Warning:* It is usually only possible to find the inverse if the function is monotonic (see section 1.3). Otherwise, if you know the output, you can't be sure what the input was. For example, think about the function $y = x^2$. If we know that the output is 9, say, we can't tell whether the input was 3 or -3. In such cases, we say that the inverse function "doesn't exist."

- There is an easy way to work out what the graph of the inverse function looks like: just reflect it in the line $y = x$.



> Exercises 4.6: Find the inverse of each of the following functions:
> (1) $f(x) = 8x + 7$
> (2) $g(x) = 3 - 0.5x$
> (3) $h(x) = \frac{1}{x+4}$
> (4) $k(x) = x^3$

Further reading and exercises
- *Jacques* §1.3.
- *Anthony & Biggs* §2.2.
- A-level pure maths textbooks.

4. Economic Application: Supply and Demand Functions

4.1. Demand

The market demand function for a good tells us how the quantity that consumers want to buy depends on the price. Suppose the demand function in a market is:

$$Q^d(P) = 90 - 5P$$

This is a linear function of P. You can see that it is downward-sloping (it is a decreasing function). If the price increases, consumers will buy less.

The inverse demand function tells us how much consumers will pay if the quantity available is Q. To find the inverse demand function:

$$\begin{aligned} Q &= 90 - 5P \\ 5P &= 90 - Q \\ P &= \frac{90 - Q}{5} \end{aligned}$$

So the inverse demand function is:

$$P^d(Q) = \frac{90 - Q}{5}$$

4.2. Supply

Suppose the supply function (the quantity that firms are willing to supply if the market price is P) is:

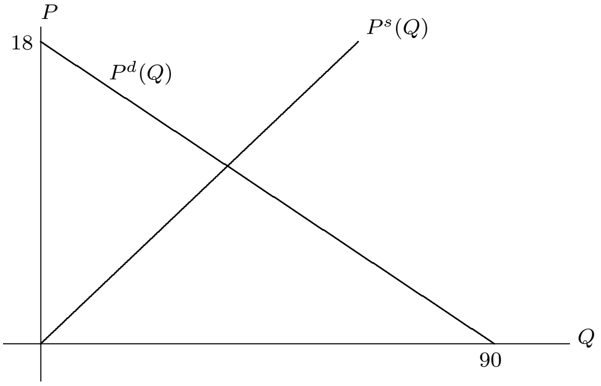
$$Q^s(P) = 4P$$

This is an increasing function (upward-sloping). Firms will supply more if the price is higher. The inverse supply function is:

$$P^s(Q) = \frac{Q}{4}$$

4.3. Market Equilibrium

We can draw the supply and demand functions to show the equilibrium in the market. It is conventional to draw the graph of P against Q – that is, to put P on the vertical axis, and draw the *inverse* supply and demand functions).



The equilibrium in the market is where the supply price equals the demand price:

$$\begin{aligned} P^s(Q) &= P^d(Q) \\ \Longrightarrow \frac{Q}{4} &= \frac{90 - Q}{5} \\ 5Q &= 360 - 4Q \\ Q &= 40 \end{aligned}$$

and so:

$$P = 10$$

EXERCISES 4.7: Suppose that the supply and demand functions in a market are:

$$Q^s(P) = 6P - 10$$
$$Q^d(P) = \frac{100}{P}$$

(1) Find the inverse supply and demand functions, and sketch them.

(2) Find the equilibrium price and quantity in the market.

4.4. Using Parameters to Specify Functions

In economic applications, we often want to specify the general shape of a function, without giving its exact formula. To do this, we can include *parameters* in the function.

For example, in the previous section, we used a particular demand function:

$$P^d(Q) = \frac{90 - Q}{5}$$

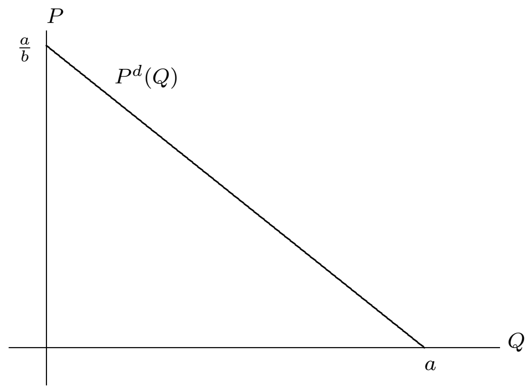
A more general specification would be to write it using two parameters a and b, instead of the numbers:

$$P^d(Q) = \frac{a - Q}{b} \text{ where } a > 0 \text{ and } b > 0$$

This gives us enough information to sketch the general shape of the function:

- it is a downward-sloping straight line (if we write it as
$$P = -\frac{1}{b}Q + \frac{a}{b}$$
we can see that the gradient is negative);

- and we can find the points where it crosses the axes.



EXERCISES 4.8: Suppose that the inverse supply and demand functions in a market are:

$$P^d(Q) = a - Q$$
$$P^s(Q) = cQ + d \quad \text{where } a, c, d > 0$$

(1) Sketch the functions.

(2) Find the equilibrium quantity in terms of the parameters a, c and d.

(3) What happens if $d > a$?

Further reading and exercises

- *Jacques* §1.3.
- *Anthony & Biggs* §1.

5. Exponential and Logarithmic Functions

5.1. Exponential Functions

$$f(x) = a^x$$

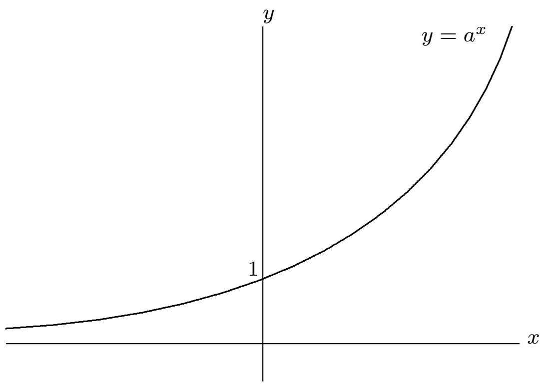
where a is any positive number, is called an *exponential function.* For example, 3^x and 8.31^x are both exponential functions.

Exponential functions functions have the same general shape, whatever the value of a. If $a > 1$:

- the y-intercept is 1, because $a^0 = 1$;
- y is positive, and increasing, for all values of x;
- as x gets bigger, y increases very fast (exponentially);
- as x gets more negative, y gets closer to zero (but never actually gets there):

$$\lim_{x \to -\infty} a^x = 0$$

(If $a < 1$, a^x is a decreasing function.)



5.2. Logarithmic Functions

$$g(x) = \log_a x$$

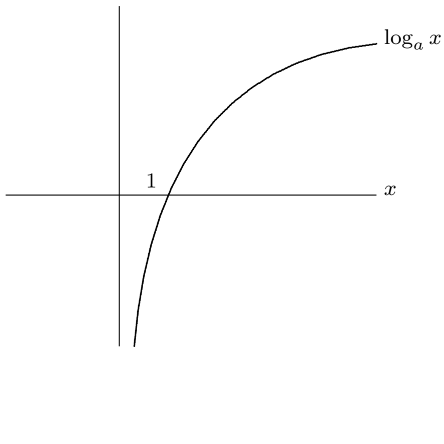
is a logarithmic function, with base a. Remember, from the definition of a logarithm in Chapter 1, that:

$$z = \log_a x \quad \text{is equivalent to} \quad x = a^z$$

In other words, a logarithmic function is the inverse of an exponential function.

Since a logarithmic function is the inverse of an exponential function, we can find the shape of the graph by reflecting the graph above in the line $y = x$. We can see that if $a > 1$:

- Since a^z is always positive, the logarithmic function is only defined for positive values of x.
- It is an increasing function
- Since $a^0 = 1$, $\log_a(1) = 0$
- $\log_a x < 0$ when $0 < x < 1$
- $\log_a x > 0$ when $x > 1$
- $\lim_{x \to 0} \log_a x = -\infty$



EXERCISES 4.9: **Exponential and Logarithmic Functions**

(1) Plot the graph of the exponential function $f(x) = 2^x$, for $-3 \le x \le 3$.

(2) What is (a) $\log_2 2$ (b) $\log_a a$?

(3) Plot the graph of $y = \log_{10} x$ for values of x between 0 and 10.
Hint: The "log" button on most calculators gives you $\log_{10}$.

5.3. *The* Exponential Function

In Chapter 3 we came across the number e: $e \approx 2.71828$. The function

$$e^x$$

is known as *the* exponential function.

5.4. Natural Logarithms

The inverse of the exponential function is the *natural logarithm function*. We could write it as:

$$\log_e x$$

but it is often written instead as:

$$\ln x$$

Note that since e^x and ln are inverse functions:

$$\ln(e^x) = x \text{ and } e^{\ln x} = x$$

5.5. Where the Exponential Function Comes From

Remember (or look again at Chapter 3) where e comes from: it is the limit of a sequence of numbers. For any number r:

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n$$

So, we can think of the exponential function as the limit of a sequence of functions:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

EXERCISES 4.10: **Base** e

(1) Plot the graphs of $1 + x$, $\left(1 + \frac{x}{2}\right)^2$, $\left(1 + \frac{x}{3}\right)^3$, and e^x, for $0 \le x \le 1$.

(2) What is (i) $\ln 1$ (ii) $\ln e$ (iii) $\ln\left(e^{5x}\right)$ (iv) $e^{\ln x^2}$ (v) $e^{\ln 3 + \ln x}$?

(3) Plot the graph of $y = \ln x$ for values of x between 0 and 3.

(4) Sketch the graph of $y = e^x$. From this sketch, work out how to sketch the graphs of $y = e^{-x}$ and $y = e^{3x+1}$.
Hint: There is an *ln* button, and an e^x button, on most calculators.

Further reading and exercises

- *Jacques* §2.4.
- *Anthony & Biggs* §7.1 to §7.4.

6. Economic Examples using Exponential and Logarithmic Functions

From Chapter 3 we know that an initial amount A_0 invested at interest rate i with continuous compounding of interest would grow to be worth

$$A(t) = A_0 e^{it}$$

after t years. We can think of this as an exponential function of time. Exponential functions are used to model the growth of other economic variables over time:

EXAMPLES 6.1: A company selling cars expects to increase its sales over future years. The number of cars sold per day after t years is expected to be:

$$S(t) = 5e^{0.08t}$$

(i) How many cars are sold per day now?
When $t = 0$, $S = 5$.

(ii) What is the expected sales rate after (a) 1 year; (b) 5 years (c) 10 years?
$S(1) = 5e^{0.08} = 5.4$.
Similarly, (b) $S(5) = 7.5$ and (c) $S(10) = 11.1$.

(iii) Sketch the graph of the sales rate against time.

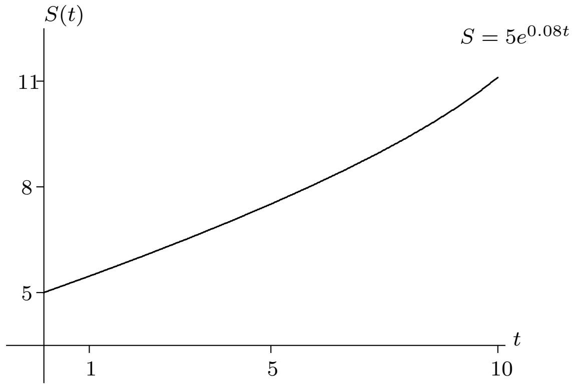
(iv) When will daily sales first exceed 14?

To find the time when $S = 14$, we must solve the equation:

$$14 = 5e^{0.08t}$$

Take (natural) logs of both sides:

$$\begin{aligned} \ln 14 &= \ln 5 + \ln(e^{0.08t}) \\ \ln 14 - \ln 5 &= 0.08t \\ 1.03 &= 0.08t \\ t &= 12.9 \end{aligned}$$



EXAMPLES 6.2: Y is the GDP of a country. GDP in year t satisfies (approximately) the equation:

$$Y = ae^{0.03t}$$

(i) What is GDP in year 0?
Putting $t = 0$ in the equation, we obtain: $Y(0) = a$. So the parameter a represents the initial value of GDP.

(ii) What is the percentage change in GDP between year 5 and year 6?
$Y(5) = ae^{0.15} = 1.1618a$ and $Y(6) = ae^{0.18} = 1.1972a$

So the percentage increase in GDP is:

$$100 \times \frac{Y(6) - Y(5)}{Y(5)} = 100\frac{.0354a}{1.1618a} = 3.05$$

The increase is approximately 3 percent.

(iii) What is the percentage change in GDP between year t and year $t+1$?

$$\begin{aligned} 100 \times \frac{Y(t+1) - Y(t)}{Y(t)} &= 100\frac{ae^{0.03(t+1)} - ae^{0.03t}}{ae^{0.03t}} \\ &= 100\frac{ae^{0.03t}(e^{0.03} - 1)}{ae^{0.03t}} \\ &= 100(e^{0.03} - 1) \\ &= 3.05 \end{aligned}$$

So the percentage increase is approximately 3 percent in *every* year.

(iv) Show that the graph of $\ln Y$ against time is a straight line.

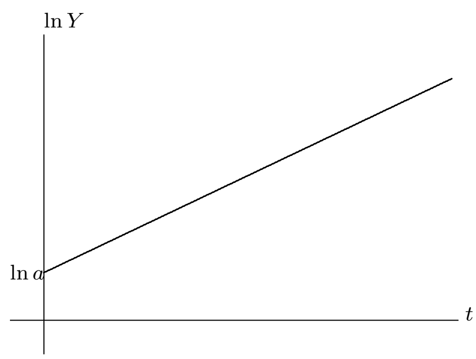
The equation relating Y and t is:

$$Y = ae^{0.03t}$$

Take logs of both sides of this equation:

$$\begin{aligned} \ln Y &= \ln(ae^{0.03t}) \\ &= \ln a + \ln(e^{0.03t}) \\ &= \ln a + 0.03t \end{aligned}$$

So if we plot a graph of $\ln Y$ against t, we will get a straight line, with gradient 0.03, and vertical intercept $\ln a$.



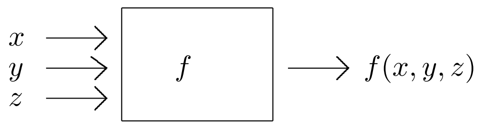
> **Exercises 4.11: Economic Example using the Exponential Function**
> The percentage of a firm's workforce who know how to use its computers increases over time according to:
>
> $$P = 100\left(1 - e^{-0.5t}\right)$$
>
> where t is the number of years after the computers are introduced.
>
> (1) Calculate P for $t = 0, 1, 5$ and 10, and hence sketch the graph of P against t.
>
> (2) What happens to P as $t \to \infty$?
>
> (3) How long is it before 95% of the workforce know how to use the computers?
> *Hint*: You can use the method that we used in the first example, but you will need to rearrange your equation before you take logs of both sides.

Further reading and exercises

- *Jacques* §2.4.

7. Functions of Several Variables

A function can have more than one input:



Examples 7.1: *Functions of Several Variables*

(i) For the function $F(x, y) = x^2 + 2y$
  (1) $F(3, 2) = 3^2 + 4 + 13$
  (2) $F(-1, 0) = (-1)^2 + 0 = 1$

(ii) The production function of a firm is: $Y(K, L) = 3K^{0.4}L^{0.6}$
In this equation, Y is the number of units of output produced with K machines and L workers.

  (1) How much output is produced with one machine and 4 workers?
  $Y(1, 4) = 3 \times 4^{0.6} = 6.9$

  (2) If the firm has 3 machines, how many workers does it need to produce 10 units of output?
  When the firm has 3 machines and L workers, it produces:

$$\begin{aligned} Y &= 3 \times 3^{0.4} L^{0.6} \\ &= 4.66 L^{0.6} \end{aligned}$$

  So if it wants to produce 10 units we need to find the value of L for which:

$$\begin{aligned} 10 &= 4.66 L^{0.6} \\ \Longrightarrow L^{0.6} &= 2.148 \\ \Longrightarrow L &= 2.148^{\frac{1}{0.6}} \\ &= 3.58 \end{aligned}$$

  It needs 3.58 (possibly 4) workers.[1]

Exercises 4.12: **Functions of Several Variables**

(1) For the function $f(x, y, z) = 2x + 6y - 7 + z^2$ evaluate (i) $f(0, 0, 0)$ (ii) $f(5, 3, 1)$

(2) A firm has production function: $Y(K, L) = 4K^2L^3$, where K is the number of units of capital, and L is the number of workers, that it employs. How much output does it produce with 2 workers and 3 units of capital? If it has 5 units of capital, how many workers does it need to produce 6400 units of output?

[1] If you are unsure about manipulating indices as we have done here, refer back to Chapter 1.

7.1. Drawing Functions of Two Variables: Isoquants

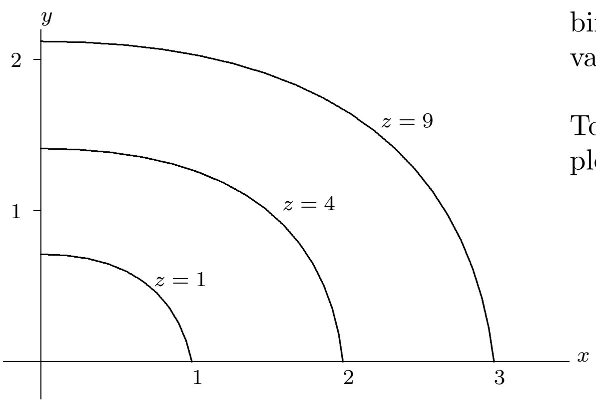
Suppose we have a function of two variables:

$$z = f(x, y)$$

To draw a graph of the function, you could think of (x, y) as a point on a horizontal plane, like the co-ordinates of a point on a map, and z as a distance above the horizontal plane, corresponding to the height of the land at the point (x, y). So the graph of the function will be a "surface" in 3-dimensions. See *Jacques* §5.1 for an example of a picture of a surface.

Since you really need 3 dimensions, graphs of functions of 2 variables are difficult to draw (and graphs of functions of 3 or more variables are impossible). But for functions of two variables we can draw a diagram like a contour map:

EXAMPLES 7.2: $z = x^2 + 2y^2$



The lines are *isoquants*: they show all the combinations of x and y that produce a particular value of the function, z.

To draw the isoquant for $z = 4$, for example:

- Write down the equation:
$$x^2 + 2y^2 = 4$$
- Make y the subject:
$$y = \sqrt{\frac{4 - x^2}{2}}$$
- Draw the graph of y against x.

7.2. Economic Application: Indifference curves

Suppose that a consumer's preferences over 2 goods are represented by the utility function:

$$U(x_1, x_2) = x_1^3 x_2^2$$

In this case an isoquant is called an *indifference curve*. It shows all the bundles (x_1, x_2) that give the same amount of utility: k, for example. So an indifference curve is represented by the equation:

$$x_1^3 x_2^2 = k$$

where k is a constant. To draw the indifference curves, make x_2 the subject:

$$x_2 = \left(\frac{k}{x_1^3}\right)^{\frac{1}{2}}$$

Then you can sketch them for different values of k (putting x_2 on the vertical axis).

EXERCISES 4.13: Sketch the indifference curves for a consumer whose preferences are represented by the utility function: $U(x_1, x_2) = x_1^{0.5} + x_2$

Further reading and exercises

- *Jacques* §5.1.
- *Anthony & Biggs* §11.1.

8. Homogeneous Functions, and Returns to Scale

EXERCISES 4.14: A firm has production function $Y(K, L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$.

(1) How much output does it produce with

(1) 2 workers and 5 units of capital?

(2) 4 workers and 10 units of capital?

(3) 6 workers and 15 units of capital?

In this exercise, you should have found that when the firm doubles its inputs, it doubles its output, and when it trebles the inputs, it trebles the output. That is, the firm has constant return to scale. In fact, for the production function

$$Y(K, L) = 4K^{\frac{1}{3}}L^{\frac{2}{3}}$$

if the inputs are multiplied by any positive number λ:

$$\begin{aligned} Y(\lambda K, \lambda L) &= 4(\lambda K)^{\frac{1}{3}}(\lambda L)^{\frac{2}{3}} \\ &= 4\lambda^{\frac{1}{3}}K^{\frac{1}{3}}\lambda^{\frac{2}{3}}L^{\frac{2}{3}} = \lambda^{\frac{1}{3}+\frac{2}{3}}4K^{\frac{1}{3}}L^{\frac{2}{3}} = \lambda 4K^{\frac{1}{3}}L^{\frac{2}{3}} \\ &= \lambda Y(K, L) \end{aligned}$$

– the output is multiplied by λ. We say that the function is *homogeneous of degree* 1.

A function $f(x, y)$ is said to be homogeneous of degree n if:
$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$
for all positive numbers λ.

We can extend this for functions of more than 2 variables in the obvious way.

EXAMPLES 8.1: *Homogeneous Functions*

(i) $f(x, y, z) = 2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}$
For this function:

$$\begin{aligned} f(\lambda x, \lambda y, \lambda z) &= 2(\lambda x)^{\frac{1}{2}} + 3(\lambda y)^{\frac{1}{2}} + (\lambda z)^{\frac{1}{2}} \\ &= 2\lambda^{\frac{1}{2}}x^{\frac{1}{2}} + 3\lambda^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda^{\frac{1}{2}}z^{\frac{1}{2}} \\ &= \lambda^{\frac{1}{2}}\left(2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}}\right) = \lambda^{\frac{1}{2}}f(x, y, z) \end{aligned}$$

So it is homogeneous of degree $\frac{1}{2}$.

(ii) The function $g(x_1, x_2) = x_1 + x_2^3$ is not homogeneous because:

$$g(\lambda x_1, \lambda x_2) = \lambda x_1 + (\lambda x_2)^3 = \lambda x_1 + \lambda^3 x_2^3$$

which cannot be written in the required form.

(iii) The production function $F(K,L) = 3KL$ is homogeneous of degree 2:

$$F(\lambda K, \lambda L) = 3\lambda K \lambda L = \lambda^2 F(K,L)$$

This means that if $K$and L are *increased* by the same factor $\lambda > 1$, then output increases by more:

$$F(\lambda K, \lambda L) = \lambda^2 F(K,L) > \lambda F(K,L)$$

So this production function has *increasing returns to scale.*

From examples like this we can see that:

A production function that is homogeneous of degree n has:
constant returns to scale if $n = 1$
increasing returns to scale if $n > 1$
decreasing returns to scale if $n < 1$

Exercises 4.15: **Homogeneous Functions**

(1) Determine whether each of the following functions is homogeneous, and if so, of what degree:
  (a) $f(x,y) = 5x^2 + 3y^2$
  (b) $g(z,t) = t(z+1)$
  (c) $h(x_1,x_2) = x_1^2(2x_2^3 - x_1^3)$
  (d) $F(x,y) = 8x^{0.7}y^{0.9}$

(2) Show that the production function $F(K,L) = aK^c + bL^c$ is homogeneous. For what parameter values does it have constant, increasing and decreasing returns to scale?

Further reading and exercises

- *Jacques* §2.3.
- *Anthony & Biggs* §12.4.

Solutions to Exercises in Chapter 4

EXERCISES 4.1:

(1) $f(2) = 5$, and $f(-4) = 17$
(2) $g(x) = 0 \Rightarrow x = 1$
(3) $C(4) = 44$

EXERCISES 4.2:

(1) (a) $f(x) = x(x-1)(x-2) \Rightarrow f(0) = 0, f(1) = 0, f(2) = 0$
 (b) $f(x) < 0,\ x < 0$
 $f(x) > 0,\ 0 < x < 1$
 $f(x) < 0,\ 1 < x < 2$
 $f(x) > 0,\ x > 2$
(2) (a) 4
 (b) $g(x) = (x^2 - 1)(4 - x^2) \Rightarrow$
 $g(-2) = 0$
 $g(-1) = 0$
 $g(1) = g(2) = 0$
 (c) $x < -2 : g(x) < 0$
 $-2 < x < -1 : g(x) > 0$
 $-1 < x < 1 : g(x) < 0$
 $1 < x < 2 : g(x) > 0$
 $x > 2 : g(x) < 0$

EXERCISES 4.3:

EXERCISES 4.4:

(1) (a) $\lim_{x\to\infty} f(x) = -\infty$
 (b) $\lim_{x\to-\infty} f(x) = -\infty$
 (c) $\lim_{x\to 0} f(x) = 2$
(2) Decreasing, $\lim_{x\to\infty} y = 3$
(3) (a) Increasing function.
 (b) $\lim_{x\to\infty} g(x) = 1$
 (c) $\lim_{x\to 0} g(x) = -\infty$

EXERCISES 4.5:

(1) $f(g(2)) = -27$, $g(f(2)) = -9$
(2) $g(h(1)) = \frac{4}{3}$, $h(g(1)) = 5$
(3) $k(m(3)) = 3$, $m(k(3)) = 3$. *Cube root* is the inverse of *cube*.
(4) $g(f(x)) = 2(x+1)^2$
 $f(g(x)) = 2x^2 + 1$
(5) $h(k(x)) = \frac{5x}{2x+1}$, $k(h(x)) = \frac{x+2}{5}$

EXERCISES 4.6:

(1) $f^{-1}(y) = \frac{y-7}{8}$
(2) $g^{-1}(y) = 6 - 2y$
(3) $h^{-1}(y) = \frac{1-4y}{y} = \frac{1}{y} - 4$
(4) $k^{-1}(y) = \sqrt[3]{x}$

EXERCISES 4.7:

(1) $P^S = \frac{Q+10}{6}$, $P^D = \frac{100}{Q}$
(2) $Q = 20$, $P = 5$

EXERCISES 4.8:

(1)
(2) $Q = \frac{a-d}{c+1}$, $P = \frac{ac+d}{c+1}$.
(3) $d > a \Rightarrow$ there is no equilibrium in the market.

EXERCISES 4.9:

(1)
(2) (a) 1
 (b) 1

EXERCISES 4.10:

(1)
(2) (a) 0
 (b) 1
 (c) $5x$
 (d) x^2
 (e) $3x$

EXERCISES 4.11:

(1) $t = 0$, $P = 0$
 $t = 1$, $P \approx 39.35$
 $t = 5$, $P \approx 91.79$
 $t = 10$, $P \approx 99.33$
(2) $\lim_{t\to\infty} P = 100$
(3) $t \approx 6$

EXERCISES 4.12:

(1) $f(0,0,0) = -7$, $f(5,3,1) = 22$
(2) $Y(3,2) = 288$, $L = 4$

EXERCISES 4.13:

EXERCISES 4.14:

(1) $Y(5,2) \approx 10.86$
(2) $Y(10,4) \approx 21.72$
(3) $Y(15,6) \approx 32.57$

EXERCISES 4.15:

(1) (a) 2
 (b) No
 (c) 5
 (d) 1.6
(2) $F(\lambda K, \lambda L) = a\lambda^c K^c + b\lambda^c L^c = \lambda^c F(K, L)$. Increasing $c > 1$, constant $c = 1$, decreasing $c < 1$

[2]This Version of Workbook Chapter 4: September 25, 2006

Worksheet 4: Functions

Quick Questions

(1) If $f(x) = 2x - 5$, $g(x) = 3x^2$ and $h(x) = \frac{1}{1+x}$:
  (a) Evaluate: $h\left(\frac{1}{3}\right)$ and $g(h(2))$
  (b) Solve the equation $h(x) = \frac{3}{4}$
  (c) Find the functions $h(f(x))$, $f^{-1}(x)$, $h^{-1}(x)$ and $f(g(x))$.

(2) A country's GDP grows according to the equation $Y(t) = Y_0 e^{gt}$.
(Y_0 and g are parameters; g is the growth rate.)
  (a) What is GDP when $t = 0$?
  (b) If $g = 0.05$, how long will it take for GDP to double?
  (c) Find a formula for the time taken for GDP to double, in terms of the growth rate g.

(3) Consider the function: $g(x) = 1 - e^{-x}$.
  (a) Evaluate $g(0)$, $g(1)$ and $g(2)$.
  (b) Is it an increasing or a decreasing function?
  (c) What is $\lim_{x\to\infty} g(x)$?
  (d) Use this information to sketch the function for $x \geq 0$.

(4) The inverse supply and demand functions for a good are: $P^s(Q) = 1 + Q$ and $P^d(Q) = a - bQ$, where a and b are parameters. Find the equilibrium quantity, in terms of a and b. What conditions must a and b satisfy if the equilibrium quantity is to be positive?

(5) Are the following functions homogeneous? If so, of what degree?
(a) $g(z, t) = 2t^2 z$ (b) $h(a, b) = \sqrt[3]{a^2 + b^2}$

Longer Questions

(1) The supply and demand functions for beer are given by:

$$q^s(p) = 50p \quad \text{and} \quad q^d(p) = 100\left(\frac{12}{p} - 1\right)$$

  (a) How many bottles of beer will consumers demand if the price is 5?
  (b) At what price will demand be zero?
  (c) Find the equilibrium price and quantity in the market.
  (d) Determine the inverse supply and demand functions $p^s(q)$ and $p^d(q)$.
  (e) What is $\lim_{q\to\infty} p^d(q)$?
  (f) Sketch the inverse supply and demand functions, showing the market equilibrium.

The technology for making beer changes, so that the unit cost of producing a bottle of beer is 1, whatever the scale of production. The government introduces a tax on the production of beer, of t per bottle. After these changes, the demand function remains the same, but the new inverse supply function is:

$$p^s(q) = 1 + t$$

(g) Show the new supply function on your diagram.
(h) Find the new equilibrium price and quantity in the market, in terms of the tax, t.
(i) Find a formula for the total amount of tax raised, in terms of t, and sketch it for $0 \leq t \leq 12$. Why does it have this shape?

(2) A firm has m machines and employs n workers, some of whom are employed to maintain the machines, and some to operate them. One worker can maintain 4 machines, so the number of production workers is $n - \frac{m}{4}$. The firm's produces:

$$Y(n, m) = \left(n - \frac{m}{4}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

units of output provided that $n > \frac{m}{4}$; otherwise it produces nothing.

(a) Show that the production function is homogeneous. Does the firm have constant, increasing or decreasing returns to scale?

(b) In the short-run, the number of machines is fixed at 8.
   (i) What is the firm's short-run production function $y(n)$?
   (ii) Sketch this function. Does the firm have constant, increasing, or decreasing returns to labour?
   (iii) Find the inverse of this function and sketch it. Hence or otherwise determine how many workers the firm needs if it is to produce 16 units of output.

(c) In the long-run, the firm can vary the number of machines.
   (i) Draw the isoquant for production of 16 units of output.
   (ii) What happens to the isoquants when the number of machines gets very large?
   (iii) Explain why this happens. Why would it not be sensible for the firm to invest in a large number of machines?