CHAPTER 5

Differentiation

Differentiation is a technique that enables us to find out how a function changes when its argument changes. It is an essential tool in economics. If you have done A-level maths, much of the maths in this chapter will be revision, but the economic applications may be new. We use the **first and second derivatives** to work out the shapes of simple functions, and find **maximum and minimum points**. These techniques are applied to **cost, production and consumption functions**. **Concave and convex functions** are important in economics.

1. What is a Derivative?

1.1. Why We Are Interested in the Gradient of a Function

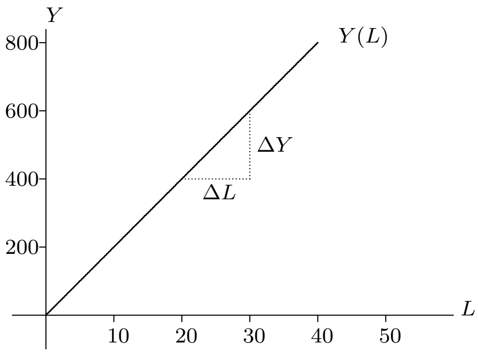
In many economic applications we want to know how a function changes when its argument changes – so we need to know its gradient.

This graph shows the *production function* $Y(L)$ of a firm: if it employs L workers it produces Y units of output.

$$\text{gradient} = \frac{\Delta Y}{\Delta L} = \frac{200}{10} = 20$$

The gradient represents the *marginal product of labour* - the firm produces 20 more units of output for each extra worker it employs.

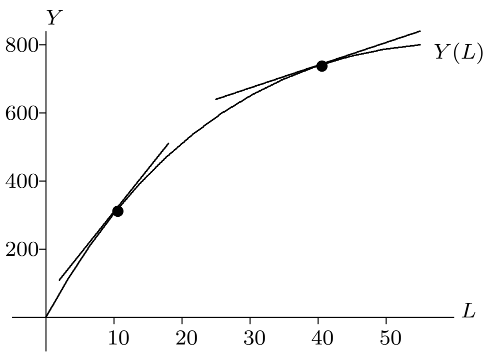
The steeper the production function, the greater is the marginal product of labour.



With this production function, the gradient changes as we move along the graph.

The gradient of a curve at a particular point is the gradient of the *tangent.*

The gradient, and hence the marginal product of labour, is higher when the firm employs 10 workers than when it employs 40 workers.



1.2. Finding the Gradient of a Function

If we have a linear function: $y(x) = mx + c$

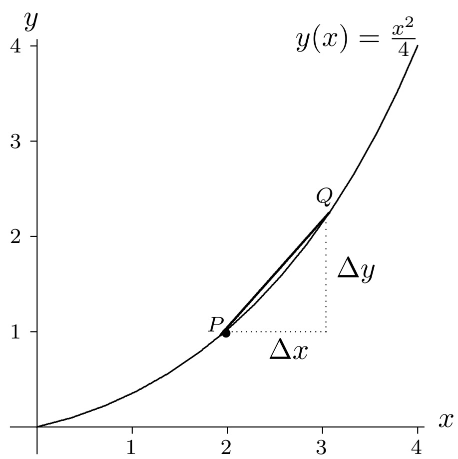
we know immediately that its graph is a straight line, with gradient equal to m (Chapter 2).

But the graph of the function: $y(x) = \dfrac{x^2}{4}$

is a curve, so its gradient changes. To find the gradient at a particular point, we could try to draw the graph accurately, draw a tangent, and measure its gradient. But we are unlikely to get an accurate answer. Alternatively:

EXAMPLES 1.1: For the function $y(x) = \frac{x^2}{4}$

(i) What is the gradient at the point where $x = 2$?



An approximation to the gradient at P is:

$$\text{gradient of } PQ = \frac{y(3) - y(2)}{3 - 2} = 1.25$$

For a better approximation take a point nearer to P:

$$\frac{y(2.5) - y(2)}{2.5 - 2} = 1.125$$

or much nearer:

$$\frac{y(2.001) - y(2)}{2.001 - 2} = 1.00025$$

So we can see that:

- In general the gradient at P is given by: $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- In this case the gradient at P is 1.

(ii) What is the gradient at the point where at the point where $x = 3$?
Applying the same method:

$$\begin{aligned}
\frac{y(3.1) - y(3)}{3.1 - 3} &= 1.525 \\
\frac{y(3.01) - y(3)}{3.01 - 3} &= 1.5025 \\
\frac{y(3.001) - y(3)}{3.001 - 3} &= 1.50025 \\
&\cdots \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= 1.5
\end{aligned}$$

1.3. The Derivative

In the example above we found the gradient of a function by working out $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

- The shorthand for: $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ is $\dfrac{dy}{dx}$
- $\dfrac{dy}{dx}$ is pronounced "dee y by dee x"
- $\dfrac{dy}{dx}$ measures the gradient
- $\dfrac{dy}{dx}$ is called the *derivative* of y

We found the derivative of the function $y(x) = \dfrac{x^2}{4}$ at particular points:

$$\frac{dy}{dx} = 1 \quad \text{at } x = 2$$
$$\frac{dy}{dx} = 1.5 \quad \text{at } x = 3$$

EXERCISES 5.1: Use the method above to find the derivative of the function $y(x) = \frac{3}{4}x^3$ at the point where $x = 2$

Further Reading

- *Jacques* §4.1 introduces derivatives slowly and carefully.
- *Anthony & Biggs* §6.2 is brief.

2. Finding the Derivative of the Function $y = x^n$

Although we could use the method in the previous section to find the derivative of any function at any particular point, it requires tedious calculations. Instead there is a simple formula, which we can use to find the derivative of any function of the form $y = x^n$, at any point.

> If $y = x^n$, then $\dfrac{dy}{dx} = nx^{n-1}$

So the derivative of the function x^n is itself a function: nx^{n-1}
and we just have to evaluate this to find the gradient at any point.

EXAMPLES 2.1: For the function $y = x^3$:

(i) Find the derivative.

Applying the formula, with $n = 3$: $\dfrac{dy}{dx} = 3x^2$

(ii) Find the gradient at the points: $x = 1$, $x = -2$ and $x = 10$

When $x = 1$ $\quad \dfrac{dy}{dx} = 3 \times 1^2 = 3.$ So the gradient is 3.

When $x = -2$ $\quad \dfrac{dy}{dx} = 3 \times (-2)^2 = 12$

When $x = 10$ $\quad \dfrac{dy}{dx} = 3 \times (10)^2 = 300$

Some Notation: We could write this as: $\left.\dfrac{dy}{dx}\right|_{x=10} = 300$

> The process of finding the derivative function is called *differentiation*

EXAMPLES 2.2: Differentiate the following functions.

(i) $y(x) = x^9$

$$\frac{dy}{dx} = 9x^8$$

(ii) $Q(P) = Q^{\frac{1}{2}}$

$$\frac{dQ}{dP} = \frac{1}{2}Q^{-\frac{1}{2}} = \frac{1}{2Q^{\frac{1}{2}}}$$

(iii) $Y(t) = t^{1.2}$

$$\frac{dY}{dt} = 1.2t^{0.2}$$

(iv) $F(L) = L^{\frac{2}{3}}$

$$\frac{dF}{dL} = \frac{2}{3}L^{-\frac{1}{3}} = \frac{2}{3L^{\frac{1}{3}}}$$

(v) $y = \dfrac{1}{x^2}$

In this case $n = -2$:

$$y = x^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$$

We have not proved that the formula is correct, but for two special cases we can verify it:

EXAMPLES 2.3: *The special cases $n = 0$ and $n = 1$*

(i) Consider the linear function: $y = x$
We know (from Chapter 2) that it is a straight line with gradient 1.

We could write this function as: $y(x) = x^1$
and apply the formula: $\dfrac{dy}{dx} = 1 \times x^0 = 1$

So the formula also tells us that the gradient of $y = x$ is 1 for *all* values of x.

(ii) Consider the constant function: $y(x) = 1$
We know (Chapter 2) that it is a straight line with gradient zero and y-intercept 1.

We could write this function as: $y(x) = x^0$
and apply the formula: $\dfrac{dy}{dx} = 0 \times x^{-1} = 0$

So the formula also tells us that the gradient is zero.

EXERCISES 5.2: **Using the formula to find derivatives**

(1) If $y = x^4$, find $\dfrac{dy}{dx}$ and hence find the gradient of the function when $x = -2$

(2) Find the derivative of $y = x^{\frac{3}{2}}$. What is the gradient at the point where $x = 16$?

(3) If $y = x^7$ what is $\left.\dfrac{dy}{dx}\right|_{x=1}$?

(4) Find the derivative of the function $z(t) = t^2$ and evaluate it when $t = 3$

(5) Differentiate the supply function $Q(P) = P^{1.7}$

(6) Differentiate the utility function $u(y) = y^{\frac{5}{6}}$ and hence find its gradient when $y = 1$

(7) Find the derivative of the function $y = \dfrac{1}{x^3}$, and the gradient when $x = 2$.

(8) Differentiate the demand function $Q = \dfrac{1}{P}$
What happens to the derivative (a) as $P \to 0$ (b) as $P \to \infty$?

Further Reading and exercises

- *Jacques* §4.1.
- *Anthony & Biggs* §6.2.
- For more examples, use an A-level pure maths textbook.

3. Optional Section: Where Does the Formula Come From?

Remember the method we used in section 1 to find the gradient of a function $y(x)$ at the point $x = 2$. We calculated:

$$\frac{y(2+h) - y(2)}{(2+h) - 2}$$

for values of h such as 0.1, 0.01, 0.001, getting closer and closer to zero.
In other words, we found:

$$\left.\frac{dy}{dx}\right|_{x=2} = \lim_{h \to 0} \frac{y(2+h) - y(2)}{(2+h) - 2} = \lim_{h \to 0} \frac{y(2+h) - y(2)}{h}$$

More generally, we could do this for any value of x (not just 2):

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

So, for the function $y = x^2$:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

We have *proved* that if $y = x^2$, $\frac{dy}{dx} = 2x$.

This method is called *Differentiation from First Principles*. You could do it for other integer values of n to check that the formula gives the correct answer. You could also apply it to some functions that are not of the simple form $y = x^n$ as in the third exercise below.

Exercises 5.3: Using the method of Differentiation from First Principles:

(1) Prove that if $y = x^3$, $\frac{dy}{dx} = 3x^2$

(2) Prove that if $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$

(3) Find the derivative of $y = 3x^2 + 5x + 1$

Further Reading

- *Anthony & Biggs* §6.1.
- Most A-level pure maths textbooks give a general proof of the formula for the derivative of x^n, by *Differentiation from First Principles*, for integer values of n.
- The proof for fractional values is a bit harder. See, for example, *Simon & Blume*.

4. Differentiating More Complicated Functions

The simple rule for differentiating $y = x^n$ can be easily extended so that we can differentiate other functions, such as polynomials. First it is useful to have some alternative notation:

> The derivative of the function $y(x)$ can be written as
> $y'(x)$ instead of $\dfrac{dy}{dx}$

EXAMPLES 4.1: For the function $y(x) = x^2$, we can write:

Either:	$\dfrac{dy}{dx} = 2x$	and	$\left.\dfrac{dy}{dx}\right\rvert_{x=5} = 10$
Or:	$y'(x) = 2x$	and	$y'(5) = 10$

You can see that the notation $y'(x)$ (read "y prime x") emphasizes that the derivative is a function, and will often be neater, especially when we evaluate the derivative at particular points. It enables us to write the following rules neatly.

> **Some rules for differentiation**
> - If $f(x) = a$ $\quad f'(x) = 0$
> - If $f(x) = ag(x)$ $\quad f'(x) = ag'(x)$
> - If $f(x) = g(x) \pm h(x)$ $\quad f'(x) = g'(x) \pm h'(x)$

(a is a constant, and f, g and h are functions.)

The first rule is obvious: the graph of a constant function is a horizontal line with zero gradient, so its derivative must be zero for all values of x. You could prove all of these rules using the method of differentiation from first principles in the previous section.

EXAMPLES 4.2: Differentiate the following functions.

(i) $f(x) = 7$
By the first rule:
$$f'(x) = 0$$

(ii) $y(x) = 4x^2$
We know the derivative of x^2 is $2x$, so using the second rule:
$$y'(x) = 4 \times 2x = 8x$$

(iii) $z(x) = x^3 - x^4$
Here we can use the third rule:
$$z'(x) = 3x^2 - 4x^3$$

(iv) $g(x) = 6x^2 + 3x + 5$
$$g'(x) = 6 \times 2x + 3 \times 1 + 0 = 12x + 3$$

(v) $h(x) = \dfrac{7}{x^3} - 5x + 4x^3$

$$\begin{aligned} h(x) &= 7x^{-3} - 5x + 4x^3 \\ h'(x) &= 7 \times (-3x^{-4}) - 5 + 4 \times 3x^2 \\ &= -\frac{21}{x^4} - 5 + 12x^2 \end{aligned}$$

(vi) $P(Q) = 5Q^{1.2} - 4Q + 10$

$$P'(Q) = 6Q^{0.2} - 4$$

(vii) $Y(t) = k\sqrt{t}$ (where k is a parameter)

$$\begin{aligned} Y(t) &= kt^{\frac{1}{2}} \\ Y'(t) &= k \times \tfrac{1}{2}t^{-\frac{1}{2}} \\ &= \frac{k}{2t^{\frac{1}{2}}} \end{aligned}$$

(viii) $F(x) = (2x+1)(1-x^2)$
First multiply out the brackets:

$$\begin{aligned} F(x) &= 2x + 1 - 2x^3 - x^2 \\ F'(x) &= 2 - 6x^2 - 2x \end{aligned}$$

EXAMPLES 4.3:

(i) Differentiate the quadratic function $f(x) = 3x^2 - 6x + 1$, and hence find its gradient at the points $x = 0$, $x = 1$ and $x = 2$.

$f'(x) = 6x - 6$, and the gradients are $f'(0) = -6$, $f'(1) = 0$, and $f'(2) = 6$.

(ii) Find the gradient of the function $g(y) = y - \frac{1}{y}$ when $y = 1$.

$g'(y) = 1 + \dfrac{1}{y^2}$, so $g'(1) = 2$.

EXERCISES 5.4: **Differentiating more complicated functions**

(1) If $f(x) = 8x^2 - 7$, what is (a) $f'(x)$ (b) $f'(2)$ (c) $f'(0)$?

(2) Differentiate the functions (a) $u(y) = 10y^4 - y^2$ (b) $v(y) = 7y + 5 - 6y^2$

(3) If $Q(P) = \frac{20}{P} - \frac{10}{P^2}$, what is $Q'(P)$?

(4) Find the gradient of $h(z) = z^3(z+4)$ at the point where $z = 2$.

(5) If $y = t - 8t^{\frac{3}{2}}$, what is $\frac{dy}{dt}$?

(6) Find the derivative of $g(x) = 3x^2 + 2 - 8x^{-\frac{1}{4}}$ and evaluate it when $x = 1$.

(7) Differentiate $y(x) = 12x^4 + 7x^3 - 4x^2 - 2x + 8$

(8) If $F(Y) = (Y-1)^2 + 2Y(1+Y)$, what is $F'(1)$?

Further Reading and Exercises

- *Jacques* §4.2.
- A-level pure maths textbooks have lots of practice exercises.

5. Economic Applications

The derivative tells you the gradient of a function $y(x)$. That means it tells you the *rate of change* of the function: how much y changes if x increases a little. So, it has lots of applications in economics:

- How much would a firm's output increase if the firm increased employment?
- How much do a firm's costs increase if it increases production?
- How much do consumers increase their consumption if their income increases?

Each of these questions can be answered by finding a derivative.

5.1. Production Functions

If a firm produces $Y(L)$ units of output when the number of units of labour employed is L, the derivative of the production function is the marginal product of labour:

$$\text{MPL} = \frac{dY}{dL}$$

(Units of labour could be workers, or worker-hours, for example.)

Examples 5.1: Suppose that a firm has production function $Y(L) = 60L^{\frac{1}{2}}$

The marginal product of labour is: $\dfrac{dY}{dL} = 60\times \frac{1}{2}L^{-\frac{1}{2}} = \dfrac{30}{L^{\frac{1}{2}}}$

If we calculate the marginal product of labour when $L = 1$, 4, and 9 we get:

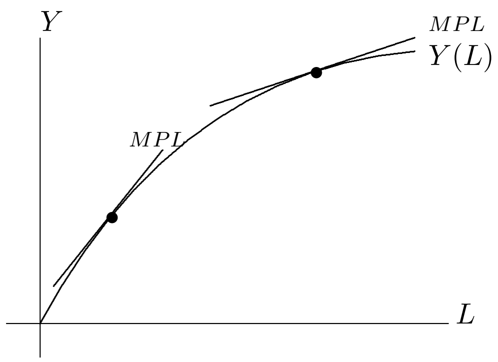
$$\left.\frac{dY}{dL}\right|_{L=1} = 30, \quad \left.\frac{dY}{dL}\right|_{L=4} = 15, \quad \left.\frac{dY}{dL}\right|_{L=9} = 10$$

These figures suggest that the firm has diminishing returns to labour.

If we look again at

$$\frac{dY}{dL} = \frac{30}{L^{\frac{1}{2}}}$$

we can see that the marginal product of labour falls as the labour input increases.



5.2. Cost Functions

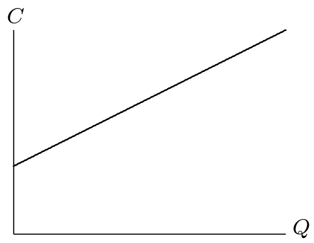
A cost function specifies the total cost $C(Q)$ for a firm of producing a quantity Q of output, so the derivative tells you how much an additional unit of output adds to costs:

The derivative of the total cost function $C(Q)$ is the marginal cost: $\text{MC} = \dfrac{dC}{dQ}$

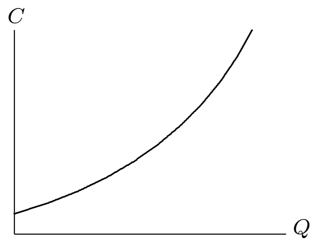
EXAMPLES 5.2: Consider the following total cost functions:

(i) $C(Q) = 4Q + 7$
Differentiating the cost function: MC=4. So marginal cost is constant for all levels of output Q. The cost function is an upward-sloping straight line.

(ii) $C(Q) = 3Q^2 + 4Q + 1$
In this case, MC=$6Q + 4$. Looking at this expression for MC, we can see that the firm has increasing marginal cost – the higher the level of output, the more it costs to make an additional unit.



(i) Constant marginal cost



(ii) Increasing marginal cost

5.3. Consumption Functions

In macroeconomics the consumption function $C(Y)$ specifies how aggregate consumption C depends on national income Y.

The derivative of the consumption function $C(Y)$ is the marginal propensity to consume:
$$\text{MPC} = \frac{dC}{dY}$$

The marginal propensity to consume tell us how responsive consumer spending would be if, for example, the government were increase income by reducing taxes.

EXERCISES 5.5: **Economic Applications**

(1) Find the marginal product of labour for a firm with production function $Y(L) = 300L^{\frac{2}{3}}$. What is the MPL when it employs 8 units of labour?

(2) A firm has cost function $C(Q) = a + 2Q^k$, where a and k are positive parameters. Find the marginal cost function. For what values of k is the firm's marginal cost (a) increasing (b) constant (c) decreasing?

(3) If the aggregate consumption function is $C(Y) = 10 + Y^{0.9}$, what is (a) aggregate consumption, and (b) the marginal propensity to consume, when national income is 50?

Further Reading and Exercises

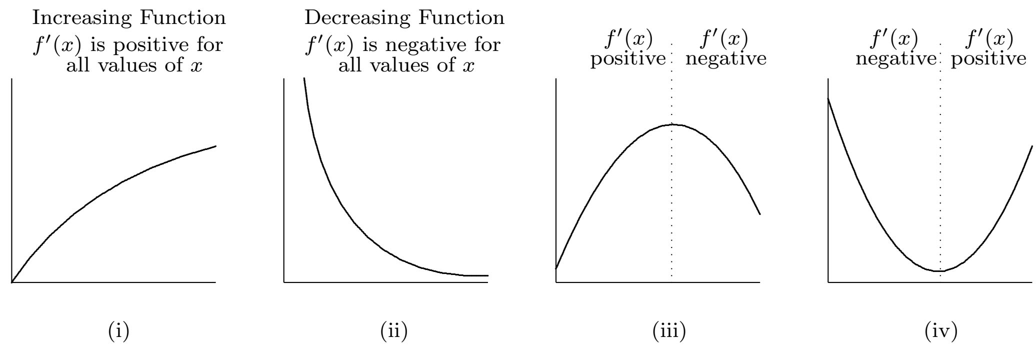
- *Jacques* §4.3.

6. Finding Stationary Points

6.1. The Sign of the Gradient

Since the derivative $f'(x)$ of a function is the gradient, we can look at the derivative to find out which way the function slopes - that is, whether the function is increasing or decreasing for different values of x.

$f'(x) > 0$	f is increasing at x
$f'(x) < 0$	f is decreasing at x



6.2. Stationary Points

In diagram (iii) there is one point where the gradient is zero. This is where the function reaches a *maximum*. Similarly in diagram (iv) the gradient is zero where the function reaches a *minimum*.

A point where $f'(x) = 0$ is called a *stationary point*, or a *turning point*, or a *critical point*, of the function.

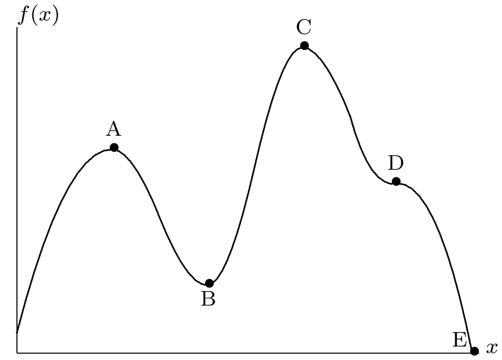
If we want to know where a function reaches a maximum or minimum, we can try to do it by looking for stationary points. But we have to be careful: not all stationary points are maxima or minima, and some maxima and minima are not stationary points. For example:

This function has four stationary points:

- a *local maximum* at A and C
- a *local minimum* at B
- and a stationary point which is neither a max or a min at D. This is called a *point of inflection*.

The *global maximum* of the function (over the range of x-values we are looking at here) is at C.

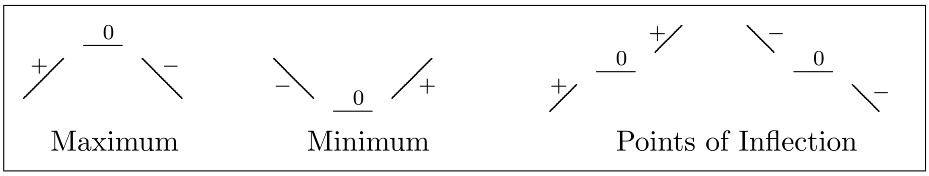
The *global minimum* of the function is at E, which is not a stationary point.



Note that there may not be a global maximum and/or minimum; we know that some functions go towards $\pm\infty$. We will see some examples below.

6.3. Identifying Maxima and Minima

- Find the stationary points of the function
- Look at the sign of the gradient on either side of the stationary points:



EXAMPLES 6.1: Find the stationary points, and the global maximum and minimum if they exist, of the following functions:

(i) $f(x) = x^2 - 2x$

The derivative is: $f'(x) = 2x - 2$
Stationary points occur where $f'(x) = 0$:

$$2x - 2 = 0 \quad \Rightarrow \quad x = 1$$

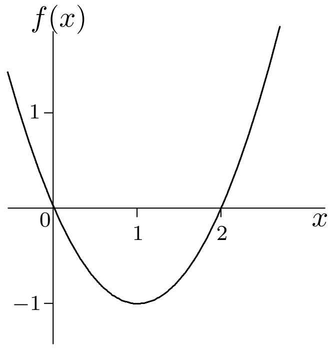
So there is one stationary point, at $x = 1$.

When $x < 1$, $f'(x) = 2x - 2 < 0$
When $x > 1$, $f'(x) = 2x - 2 > 0$

So the shape of the function at $x = 1$ is ⌣
It is a minimum point.
The value of the function there is $f(1) = -1$.

The function has no maximum points, and no global maximum.

(You can sketch the function to see exactly what it looks like)



(ii) $g(x) = 2x^3 - \frac{3}{4}x^4$

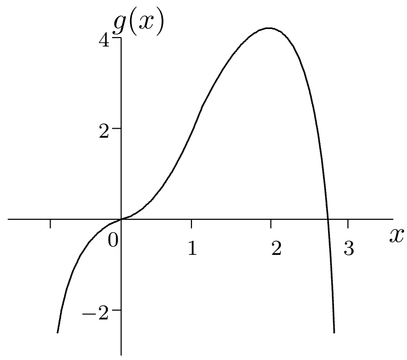
Derivative is: $g'(x) = 6x^2 - 3x^3 = 3x^2(2 - x)$

Stationary points occur where $g'(x) = 0$:

$$3x^2(2 - x) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 2$$

So two stationary points, at $x = 0$ and $x = 2$.

$$\begin{aligned} x < 0: &\quad g'(x) = 3x^2(2 - x) > 0 \\ 0 < x < 2: &\quad g'(x) > 0 \\ x > 2: &\quad g'(x) < 0 \end{aligned}$$



At $x = 0$ the gradient doesn't change sign. Point of inflection. $f(0) = 0$
At $x = 2$ the shape of the function is ⌢ . Maximum point. $f(2) = 4$
There are no minimum points, and the global minimum does not exist.

(iii) $h(x) = \dfrac{2}{x}$ for $0 < x \leq 1$

In this example we are only considering a limited range of values for x.

The derivative is: $h'(x) = -\dfrac{2}{x^2}$

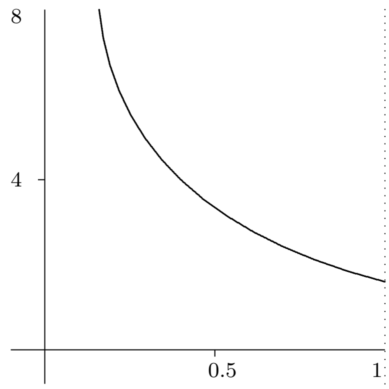
There are *no* stationary points because:

$-\dfrac{2}{x^2} < 0$ for all values of x between 0 and 1

The gradient is always negative: h is a decreasing function of x.

So the minimum value of the function is at $x = 1$: $h(1) = 2$.

The function doesn't have a maximum value because:
$h(x) \to \infty$ as $x \to 0$.



Exercises 5.6: **Stationary Points**
For each of the following functions (i) find and identify all of the stationary points, and (ii) find the global maximum and minimum values of the function, if they exist.

(1) $a(x) = x^2 + 4x + 2$

(2) $b(x) = x^3 - 3x + 1$

(3) $c(x) = 1 + 4x - x^2$ for $0 \leq x \leq 3$

(4) $d(x) = \frac{\alpha}{2}x^2 + \alpha x + 1$ where α is a parameter.
Hint: consider the two cases $\alpha \geq 0$ and $\alpha < 0$.

Further Reading and Exercises

- *Jacques* §4.6
- *Anthony & Biggs* §8.2 and §8.3

Both of these books use the second derivative to identify stationary points, so you may want to read section 7.2 before looking at them.

7. The Second Derivative

- The derivative $y'(x)$ tells us the gradient, or in other words how y changes as x increases.
- But $y'(x)$ is a function.
- So we can differentiate $y'(x)$, to find out how $y'(x)$ changes as x increases.

The derivative of the derivative is called the *second derivative*. It can be denoted $y''(x)$, but there are alternatives:

For the function $y(x)$ the derivatives can be written:

1^{st} derivative:	$y'(x)$	or	$\dfrac{d}{dx}(y(x))$	or	$\dfrac{dy}{dx}$
2^{nd} derivative:	$y''(x)$	or	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$	or	$\dfrac{d^2y}{dx^2}$

Examples 7.1: *Second Derivatives*

(i) $f(x) = ax^2 + bx + c$

The 1^{st} derivative is: $f'(x) = 2ax + b$
Differentiating again: $f''(x) = 2a$

(ii) $y = \dfrac{5}{x^2}$

This function can be written as $y = 5x^{-2}$ so:

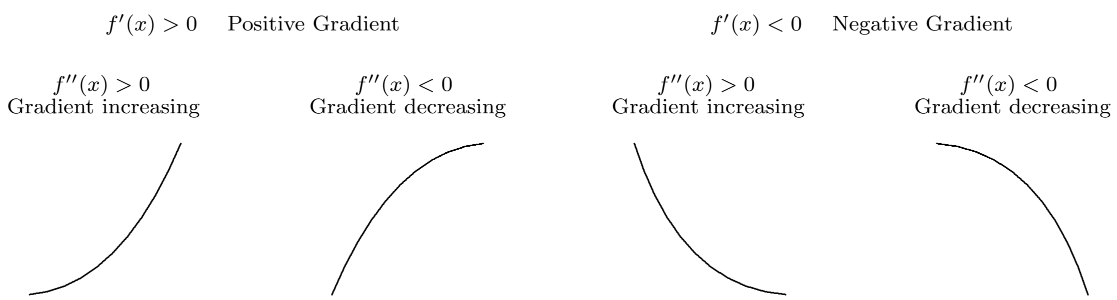
First derivative: $\dfrac{dy}{dx} = -10x^{-3} = -\dfrac{10}{x^3}$

Second derivative: $\dfrac{d^2y}{dx^2} = 30x^{-4} = \dfrac{30}{x^4}$

Exercises 5.7: Find the first and second derivatives of:
(1) $f(x) = x^4$ (2) $g(x) = x + 1$ (3) $h(x) = 4x^{\frac{1}{2}}$ (4) $k(x) = x^3(5x^2 + 6)$

7.1. Using the Second Derivative to Find the Shape of a Function

- The 1^{st} derivative tells you whether the gradient is positive or negative.
- The 2^{nd} derivative tells you whether the gradient is increasing or decreasing.



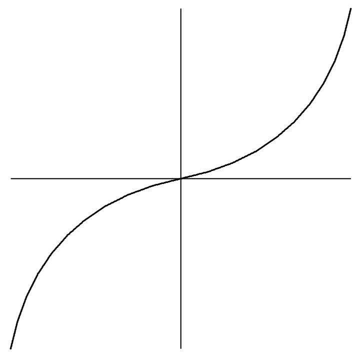
EXAMPLES 7.2: $y = x^3$

From the derivatives of this function:

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x$$

we can see:

- There is one stationary point, at $x = 0$ and at this point $y = 0$
- When $x < 0$:
  $$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} < 0$$
  The gradient is positive, and decreasing.
- When $x < 0$:
  $$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} > 0$$
  The gradient is positive, and increasing.



7.2. Using the 2^{nd} Derivative to Classify Stationary Points

In section 6.3 we looked at the gradient on either side of a stationary point to determine its type. But instead we can look at the sign of the second derivative, to see whether the gradient is increasing (as it will be at a minimum point, from negative to positive) or decreasing (as it will be at a maximum point, from positive to negative).

> If $f(x)$ has a stationary point at $x = x_0$
> (that is, if $f'(x_0) = 0$)
>
> then if $f''(x_0) > 0$ it is a minimum point
> and if $f''(x_0) < 0$ it is a maximum point

If the second derivative is zero it doesn't help you to classify the point, and you have to revert to the previous method.

EXAMPLES 7.3: Applying this method to the examples in section 6.3:

(i) $f(x) = x^2 - 2x$

$$f'(x) = 2x - 2$$

There is one stationary point at $x = 1$.

$$f''(x) = 2$$

This is positive, for all values of x. So the stationary point is a minimum.

(ii) $g(x) = 2x^3 - \frac{3}{4}x^4$

$$g'(x) = 6x^2 - 3x^3 = 3x^2(x - 2)$$

There are two stationary points, at $x = 0$ and $x = 2$.

$$g''(x) = 12x - 9x^2 = 3x(4 - 3x)$$

Look at each stationary point:

$x = 0$: $g''(0) = 0$ – we can't tell its type. (We found before that it is a point of inflection.)

$x = 2$: $g''(2) = -12$ – it is a maximum point.

EXERCISES 5.8: Find and classify all the stationary points of the following functions:

(1) $a(x) = x^3 - 3x$

(2) $b(x) = x + \dfrac{9}{x} + 7$

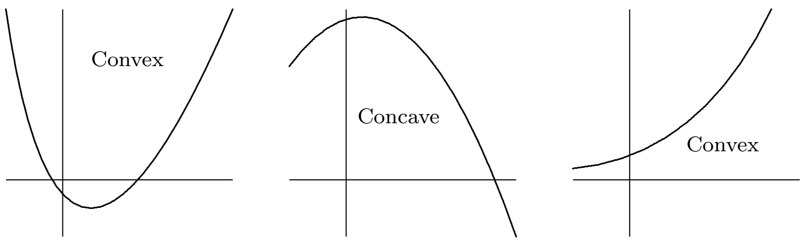
(3) $c(x) = x^4 - 4x^3 - 8x^2 + 12$

7.3. Concave and Convex Functions

If the gradient of a function is *increasing* for *all* values of x, the function is called *convex*. If the gradient of a function is *decreasing* for *all* values of x, the function is called *concave*. Since the second derivative tells us whether the gradient is increasing or decreasing:

A function f is called $\left\{ \begin{array}{c} \text{concave} \\ \text{convex} \end{array} \right\}$ if

$$f''(x) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} 0 \quad \text{for all values of } x$$



EXAMPLES 7.4: Are the following functions concave, convex, or neither?

(i) The quadratic function $f(x) = ax^2 + bx + c$.

$f''(x) = 2a$, so if $a > 0$ it is convex and if $a < 0$ it is concave.

(ii) $P(q) = \dfrac{1}{q^2}$

$P'(q) = -2q^{-3}$, and $P''(q) = 6q^{-4} = \dfrac{6}{q^4}$

The second derivative is positive for all values of q, so the function is convex.

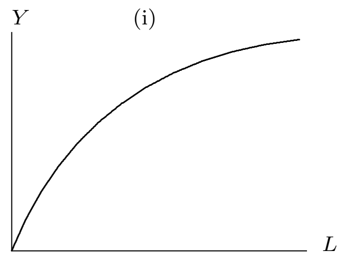
(iii) $y = x^3$

We looked at this function in section 7.1. The second derivative changes sign, so the function is neither concave nor convex.

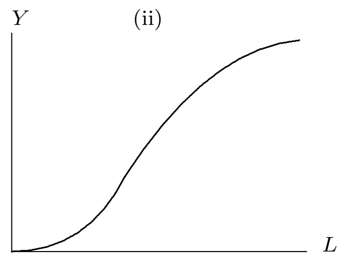
7.4. Economic Application: Production Functions

The economic characteristics of a production function $Y(L)$ depend on its shape:

– It is (usually) an increasing function: $Y'(L) > 0$ for all values of x;
– but the second derivative could be positive or negative: it tells you whether the marginal product of labour is increasing or decreasing.



– $Y''(L) < 0$ for all values of L
– The production function is concave
– MPL is decreasing
– Diminishing returns to labour



This function is neither concave nor convex.
When employment is low:

– $Y''(L) > 0$
– Increasing returns to labour

But when emplyment is higher:

– $Y''(L) < 0$
– Decreasing returns to labour

Exercises 5.9: **Concave and Convex Functions in Economics**

(1) Consider the cost function $C(q) = 3q^2 + 2q + 5$
  (a) What is the marginal cost?
  (b) Use the second derivative to find out whether the firm has increasing or decreasing marginal cost.
  (c) Is the cost function convex or concave?
(2) For the aggregate consumption function is $C(Y) = 10 + Y^{0.9}$
  (a) Find the first and second derivatives.
  (b) How does the marginal propensity to consume change as income increases?
  (c) Is the function concave or convex?
  (d) Sketch the function.

Further Reading and Exercises

- *Jacques* §4.3 on the derivatives of economic functions
  and §4.6 on maxima and minima
- *Anthony & Biggs* §8.2 and §8.3 for stationary points, maxima and minima

Solutions to Exercises in Chapter 5

EXERCISES 5.1:

(1) $\left.\frac{dy}{dx}\right|_{x=2} = 9$

EXERCISES 5.2:

(1) $\frac{dy}{dx} = 4x^3$,
$\left.\frac{dy}{dx}\right|_{x=-2} = -32$

(2) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$,
$\left.\frac{dy}{dx}\right|_{x=16} = 6$

(3) $\left.\frac{dy}{dx}\right|_{x=1} = 7$

(4) $z'(t) = 2t$, $z'(3) = 6$

(5) $Q'(P) = 1.7P^{0.7}$

(6) $u'(y) = \frac{5}{6}y^{-\frac{1}{6}}$, $u'(1) = \frac{5}{6}$

(7) $\frac{dy}{dx} = \frac{-3}{x^4}$, $\left.\frac{dy}{dx}\right|_{x=2} = -\frac{3}{16}$

(8) $\frac{dQ}{dP} = -\frac{1}{P^2}$
(a) $\lim_{P\to 0} \frac{dQ}{dP} = -\infty$
(b) $\lim_{P\to\infty} \frac{dQ}{dP} = 0$

EXERCISES 5.3:

(1) $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$

(2) $\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-x-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{x^2+xh} = \frac{-1}{x^2}$

(3) $\frac{dy}{dx} = 6x + 5$

EXERCISES 5.4:

(1) (a) $f'(x) = 16x$
(b) $f'(2) = 32$
(c) $f'(0) = 0$

(2) (a) $u'(y) = 40y^3 - 2y$
(b) $v'(y) = 7 - 12y$

(3) $Q'(P) = \frac{20}{P^3} - \frac{20}{P^2}$
$= \frac{20}{P^3}(1 - P)$

(4) $h'(2) = 80$

(5) $\frac{dy}{dt} = 1 - 12\sqrt{t}$

(6) $g'(x) = 6x + 2x^{-\frac{5}{4}}$,
$g'(1) = 8$

(7) $y'(x) =$
$48x^3 + 21x^2 - 8x - 2$

(8) $F'(1) = 6$

EXERCISES 5.5:

(1) $\text{MPL} = 200L^{-\frac{1}{3}}$,
$\text{MPL}(8) = 100$

(2) $C'(Q) = 2kQ^{k-1}$
(a) $k > 1$
(b) $k = 1$
(c) $k < 1$

(3) (a) $C(50) \approx 43.81$
(b) $C'(Q) \approx 0.61$

EXERCISES 5.6:

(1) $x = -2, a(-2) = -2$: global minimum.

(2) $x = -1, b(-1) = 3$: maximum point.
$x = 1, b(1) = -1$: minimum point.
No global max or min.

(3) $x = 2, c(2) = 5$: global maximum.
No other stationary points; global minimum (in range $0 \le x \le 3$) is at $x = 0, c(0) = 1$.

(4) Stationary point at $x = 1$, $d(1) = \frac{3}{2}\alpha + 1$.
It is a global max if $\alpha < 0$ and a global min if $\alpha > 0$

EXERCISES 5.7:

(1) $f'(x) = 4x^3$
$f''(x) = 12x^2$

(2) $g'(x) = 1$
$g''(x) = 0$

(3) $h'(x) = 2x^{-\frac{1}{2}}$
$h''(x) = -x^{-\frac{3}{2}}$

(4) $k'(x) = 25x^4 + 18x^2$
$k''(x) = 100x^3 + 36x$

EXERCISES 5.8:

(1) $(-1, 2)$ max
$(1, -2)$ min

(2) $(-3, 1)$ max
$(3, 13)$ min

(3) $(-1, 9)$ min
$(0, 12)$ max
$(4, -116)$ min

EXERCISES 5.9:

(1) (a) $C'(q) = 6q + 2$
(b) $C''(q) = 6 \Rightarrow$ increasing MC
(c) convex

(2) (a) $C'(Y) = 0.9Y^{-0.1}$,
$C''(Y) = -0.09Y^{-1.1}$
(b) It decreases
(c) Concave

[1]This Version of Workbook Chapter 5: September 13, 2003

Worksheet 5: Differentiation

Quick Questions

(1) Differentiate:
   (a) $y = 9x^3 - 7x^2 + 15$
   (b) $f(x) = \dfrac{3}{4x^2}$
   (c) $Y(t) = 100t^{1.3}$
   (d) $P(Q) = Q^2 - 4Q^{\frac{1}{2}}$

(2) Determine whether each of the following functions is concave, convex, or neither:
   (a) $y = 5x^2 - 8x + 7$
   (b) $C(y) = \sqrt{4y}$ (for $y \geq 0$)
   (c) $P(q) = q^2 - 4q^{\frac{1}{2}}$ (for $q \geq 0$)
   (d) $k(x) = x^2 - x^3$

(3) Find the first and second derivatives of the production function $F(L) = 100L + 200L^{\frac{2}{3}}$ (for $L \geq 0$) and hence determine whether the firm has decreasing, constant, or increasing returns to labour.

(4) Find and classify all the the stationary points of the following functions. Find the global maxima and minima, if they exist, and sketch the functions.
   (a) $y = 3x - x^2 + 4$ for values of x between 0 and 4.
   (b) $g(x) = 6x^{\frac{1}{2}} - x$ for $x \geq 0$.
   (c) $f(x) = x^4 - 8x^3 + 18x^2 - 5$ (for all values of x)

(5) If a firm has cost function $C(Q) = aQ(b + Q^{1.5}) + c$, where a, b and c are positive parameters, find and sketch the marginal cost function. Does the firm have concave or convex costs?

Longer Questions

(1) The number of meals, y, produced in a hotel kitchen depends on the number of cooks, n, according to the production function $y(n) = 60n - \dfrac{n^3}{5}$
   (a) What is the marginal product of labour?
   (b) Does the kitchen have increasing, constant, or decreasing returns to labour?
   (c) Find:
      (i) the output of the kitchen
      (ii) the average output per cook
      (iii) the marginal product of labour
   when the number of cooks is 1, 5, 10 and 15.
   (d) Is this a realistic model of a kitchen? Suggest a possible explanation for the figures you have obtained.
   (e) Draw a careful sketch of the production function of the kitchen.
   (f) What is the maximum number of meals that can be produced?

(2) A manufacturer can produce economics textbooks at a cost of £5 each. The textbook currently sells for £10, and at this price 100 books are sold each day. The

manufacturer figures out that each pound decrease in price will sell ten additional copies each day.

(a) What is the demand function $q(p)$ for the textbooks?
(b) What is the cost function?
(c) What is the inverse demand function?
(d) Write down the manufacturer's daily profits as function of the quantity of textbooks sold, $\Pi(q)$.
(e) If the manufacturer now maximises profits, find the new price, and the profits per day.
(f) Sketch the profit function.
(g) If the government introduces a law that at least 80 economics textbooks must be sold per day, and the (law-abiding) manufacturer maximises profits, find the new price.